On Localic Convergence with Applications

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Abstract

Our main goal is to collate into a single document what is presently known regarding pointfree convergence. This will be done by exposing some well-known results on pointfree convergence in a much more simpler way. We will start to study the convergence and clustering of filters in frames in terms of covers and use this to characterise compact frames and some type of uniform frames. We will extend this study to a more general type of filters. We will then discuss convergence and clustering of filters on a locale, where a filter on a locale L is just a filter in the sublattice of all the sublocales of L. This convergence has many applications like characterising compact locales and also characterising sharp points which will also be studied. Finally, the latter concepts of convergence and clustering will be reconciled with the previous one.

Keywords: Uniform frame, compact locale, sublocale, convergent filter in a frame, clustered filter in a frame, general filter on a frame, $\mathbb{F}$-compact frame, convergent filter on a locale, clustered filter on a locale, sharp point.

Declaration

I declare that the above dissertation is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

Annette Flavie Ngo Babem, 27 January 2019.

Contents

Introduction

Pointfree topology is the branch of topology where the basic elements are no more points like in topological spaces but rather "open sets ". This theory had been lying for years at the intersection of lattice theory and topology and this was not so useful. But it is only in the middle of the fifties that the theory got a turning point with the work of Ehresmann in [15] and Benabou [10]. The author in [15] was then talking about some sort of "generalized topological spaces "which were later called locales. Then, there was a need to investigate if some concepts in topological spaces (convergence, clustering, normality, compactness and so on) could be imported from classical topology to locales. Authors like P. Johnstone [19], Picado and Pultr. [24] were able to synsethise these notions in their work.

The concept of convergence in pointfree topology was introduced by S.S. Hong [17] in terms of classical filters in a frame. Subsequently, B. Banaschewski and S.S. Hong introduced what they called general filters in [5]. T. Dube and O. Ighedo took a slightly different approach when introducing the concept of localic convergence in [13]. Since it has many applications such as characterising compact frames and also uniform frames, it is desirable to have in one document the various approaches to pointfree convergence. In this dissertation, the motivation is to collate into a single catalogue what is presently known regarding convergence in pointfree topology.

The main aim is to trace the history of convergence in pointfree topology, starting with the work of S.S. Hong [17]. We show how general filters of B. Banaschewski and S.S. Hong [5] form a natural generalisation of classical filters. Further, the concept of convergence of filters on a locale introduced by T. Dube and Ighedo [13] is reconciled with the earlier notions.

Our work is structured as follows: the first chapter consists of preliminaries. Here, we fix notations, define some basic concepts that are involved in this dissertation and which help to better understand this work. We talk about concepts like frames, uniform frames, locales, sublocales and study their properties. In the second chapter, we look at what is known regarding convergence and clustering in terms of classical filters in a frame. We talk about the concepts of strong clustering [14] and strong convergence ([8] and [17]) in a frame in terms of classical filters and we prove that they imply clustering and convergence in a frame respectively [14]. We characterise compact frames [17] and uniform frames in terms of convergence and clustering of classical filters. In the third chapter, we talk about convergence in a frame in terms of general filters (T-valued bounded meet-semilattice homomorphisms) as in [3]. We talk about strong convergence in a frame in terms of general filters [7] and we prove that this implies convergence. We show how general filters in [7] form a natural generalisation of classical filters in [17]. We characterise spatial, compact and uniformly paracompact uniform frames in terms of convergence of classical filters in a frame L. We talk about $\mathbb{F}$-compactness as in [7], which is just the compactness notion we get by imposing that a specific type of general filters are all convergent in a frame L. In the last chapter, we define concepts of convergence and clustering on a locale [4]. For a T_1-locale L (locale satisfying the T_1 property), we show how convergence on a frame L is a generalisation of the concept of strong convergence in [7] and then, it is also a generalisation of the concept of convergence in [6]. We characterise sharp points in a locale as in [13]. We also use convergence on locales to characterise convergence and clustering on compact locales as in [13].

1. Preliminaries

Here, we define some basic concepts in the pointfree sense that are going to come in handy in our work. We start with some definitions and results from lattice theory and end up with those in frame theory. We also agree with some notations that are used later. This first chapter, is mostly based on [24] and for lattice theory see [16] where the proofs of most of the results of this chapter are found. For general knowledge on topology and category theory, we refer to [21] and [1] respectively.

1.1 Frames and their homomorphisms

1.1.1 Definition. A **partially ordered set** or **poset** $(P, \leq)$ is a nonempty set P with a partial order $\leq$.

1.1.2 Example. *$(\mathbb{R}, \leq)$ and $(\mathbb{N}, \leq)$ are posets, where $\leq$ is the usual ordering of numbers.*

1.1.3 Definition. A **meet semi-lattice** $(L, \leq, \wedge, 1)$ is a poset $(L, \leq)$ in which any finite subset $S \subseteq L$ has an **infimum** (greatest lower bound), denoted by $\bigwedge S$. Here we have $1 = \bigwedge \varnothing$, called **the top element** of L. A **join semi-lattice** $(L, \leq, \vee, 0)$ is a poset $(L, \leq)$ in which any finite subset $F \subseteq L$ has a **supremum** (least upper bound), denoted by $\bigvee F$. Here we have $0 = \bigvee \varnothing$, called the **the bottom element** of L.

1.1.4 Definition. A **lattice** $(L, \leq, \wedge, \vee, 0, 1)$ is a poset $(L, \leq)$ such that for every finite subset $S \subseteq L$, $\bigwedge S \in L$ and $\bigvee S \in L$.

1.1.5 Remark. *By convention, for a lattice L, we set $0 = \bigvee \varnothing = \bigwedge L$ and we call it the* **bottom element** *of L, and $1 = \bigwedge \varnothing = \bigvee L$ becomes* **the top element** *of L. The lattice L is* **bounded** *if L has both the 0 and 1 elements. We will often write them as 0_L and 1_L respectively. Also, we will often write the join and meet in a lattice L as $\vee_L$ and $\wedge_L$ respectively.*

1.1.6 Definition. A **sublattice** of a lattice $(L, \leq, \wedge, \vee)$ is a subset $S \subseteq L$ such that $a \vee b \in S$ and $a \wedge b \in S$, for every $a, b \in S$.

1.1.7 Definition. A lattice $(L, \leq, \wedge, \vee)$ is said to be **distributive** if

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \forall a, b, c \in L.$$

Its equivalent is that:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \forall a, b, c \in L.$$

1.1.8 Example. *If X is a nonempty set, then the collection $(\mathcal{P}(X), \subseteq, \cap, \cup)$ of all subsets of X is a distributive lattice. It is called the* **power set** *of X.*

1.1.9 Definition. A subset S of a lattice $(L, \leq, \wedge, \vee)$ is a **downset** if $\forall a \in L, \forall s \in S, a \leq s$ implies that $a \in S$. If $\forall a \in L, \forall s \in S, s \leq a$ implies that $a \in S$, then S is an **upset**.

1.1.10 Remark. *Let* $(L, \leq, \wedge, \vee)$ *be a lattice and* $S \subseteq L$*, then* S *induces the following subsets on* L*:*

$$\downarrow S = \{\downarrow s \mid s \in S\}$$

and

$$\uparrow S = \{\uparrow s \mid s \in S\},$$

where $\uparrow s = \{x \in L \mid x \geq s\}$ *and* $\downarrow s = \{x \in L \mid x \leq s\}$.

1.1.11 Definition. A **chain** in a lattice $(L, \leq, \wedge, \vee)$ is a subset $S \subseteq L$ such that $a \leq b$ or $b \leq a, \forall a, b \in S$.

1.1.12 Definition. A lattice $(L, \leq, \wedge, \vee)$ is said to be **complete** if for every subset $S \subseteq L, \bigwedge S \in L$ and $\bigvee S \in L$.

1.1.13 Definition. A **frame** L is a complete lattice in which the following property is true :

$$a \wedge \left(\bigvee_{\gamma \in \Gamma} s_\gamma \right) = \bigvee_{\gamma \in \Gamma} (a \wedge s_\gamma) \quad \text{for any subset} \quad \{s_\gamma \mid \gamma \in \Gamma\} \subseteq L \quad \text{and} \quad \forall a \in L.$$

1.1.14 Example. *Let us consider a pair* (X, τ)*, where* X *is a nonempty set and* τ *is a subset of* $\mathcal{P}(X)$ *with the following:*

- $\varnothing, X \in \tau$.
- *If* $A, B \in \tau$*, then* $A \cap B \in \tau$.
- *If* $\{A_\gamma \mid \gamma \in \Gamma\}$ *is a collection of elements of* τ*, then* $\bigcup\limits_{\gamma \in \Gamma} A_\gamma \in \tau$.

The pair (X, τ) *thus defined is called a* **topological space** *and* τ *is a* **topology** *on* X*. The set* τ *is a frame and in general, we denote this frame by* $\mathfrak{O}X$ *(to point out the fact that it is a frame induced by the topological space* X*). The elements of* $\mathfrak{O}X$ *are called* **open sets***.* $\mathfrak{O}X$ *satisfies the following :*

- $0_{\mathfrak{O}X} = \varnothing$.
- $1_{\mathfrak{O}X} = X$.
- *finite meets are given by finite intersections (* $\wedge_{\mathfrak{O}X} = \cap$*).*
- *arbitrary joins are given by arbitrary unions (*$\bigvee_{\mathfrak{O}X} = \bigcup$*).*

1.1.15 Remark. *The two-element frame* $\mathbf{2} = \{0, 1\}$ *is the smallest frame and it is contained in every frame* L*. We consider it as* **the trivial frame***.*

1.1.16 Definition. A **subframe** M of a frame L is a nonempty subset $M \subseteq L$ such that:

- $0_M = 0_L$.

- $1_M = 1_L$.
- $a \wedge b \in M, \forall a, b \in M$.
- $\bigvee_{\gamma \in \Gamma} a_\gamma \in M,$ for any subset $\{a_\gamma \mid \gamma \in \Gamma\} \subseteq M$.

1.1.17 Definition. Let L be a frame and $M \subseteq L$. The **subframe generated** by M is the smallest subframe of L containing M. We say that M is a **generating set** if L is the smallest subframe containing M, which also means $\forall x \in L, \exists B \subseteq M$ such that $x = \bigvee B$. If $N \subseteq L$ is *closed under finite meets (meaning that whenever $a, b \in N$, then $a \wedge b \in N$), then the subframe (let us call it S) of L generated by N is the set $S = \{a \in L \mid a = \bigvee X \text{ for some } X \subseteq N\}$.*

1.1.18 Definition. For a frame L, we denote by $\mathfrak{D}L$ the **frame** of **all downsets** in L.

1.1.19 Definition. Let L and M be two frames. **A frame homomorphism** from L to M is a map $h\colon L \to M$ which satisfies the following:

- $h(u \wedge v) = h(u) \wedge h(v), \forall u, v \in L$.
- $h(\bigvee_{\gamma \in \Gamma} s_\gamma) = \bigvee_{\gamma \in \Gamma} h(s_\gamma),$ for any subset $\{s_\gamma \mid \gamma \in \Gamma\} \subseteq L$.

1.1.20 Proposition. *Let S be a meet semi-lattice and L a frame. For every meet semi-lattice homomorphism $s\colon S \to L$, there is a unique frame homomorphism $h\colon \mathfrak{D}S \to L$ such that $h\lambda_S = s$, where $\lambda_S\colon S \to \mathfrak{D}S$ is defined by $\lambda_S(a) = \downarrow a$.*

1.1.21 Definition. A **filter** F in a frame L is a nonempty subset $F \subseteq L$ satisfying the following properties :

- $0 \notin F$.
- $a \wedge b \in F, \forall a, b \in F$.
- $a \leq b$ implies that $b \in F, \forall a \in F, \forall b \in L$.

The filter F is **prime** if $a \in F$ or $b \in F$ whenever $a \vee b \in F$. It is **completely prime** if $S \cap F \neq \varnothing$ whenever $\bigvee S \in F, \forall S \subseteq L$. It is **a proper filter** if $F \neq L$. The filter F is an **ultrafilter** in L if it is a maximal filter in the sense of inclusion of all the filters in L. It means that for any proper filter G in L, $F \subseteq G$ implies that $F = G$.

1.1.22 Definition. A subset $M \subseteq L$ is a **filter basis** if it is closed under finite meets and $0 \notin M$ and then the filter it generates in L is given by

$$F = \{x \in L \mid x_1, x_2, \ldots, x_n \in M \text{ and } x \geq x_1 \wedge x_2 \wedge \ldots \wedge x_n, \text{ for some } n \in \mathbb{N}\}.$$

This also means that $\forall x \in F, \exists B \subseteq M$ such that $x = \bigvee B$.

1.1.23 Definition. A nonempty subset D of a frame L is **directed or updirected** if for any $a, b \in D, \exists c \in D$ such that $a \leq c$ and $b \leq c$.

1.1.24 Definition. Let $f\colon M \to N$ be any map between two nonempty sets M and N.

- For every subset $\varnothing \neq A \subseteq M$, we define the **image** of A by $f[A] = \{f(a) \mid a \in A\} \subseteq N$.
- For every subset $\varnothing \neq B \subseteq N$, we define the **inverse image** of B by
$$f^{-1}[B] = \{x \in M \mid f(x) \in B\} \subseteq M.$$
- The map f is **onto** if $\forall a \in N, \exists x \in M$ such that $a = f(x)$ and it is **one-one** if $\forall b, c \in M, f(b) = f(c)$ implies that $b = c$.

1.1.25 Example. *Let X and Y be two topological spaces and f be a continuous map $f\colon X \to Y$, then the inverse image of any open set in $\mathfrak{O}Y$ is open in $\mathfrak{O}X$.*

Hence, the map $h\colon \mathfrak{O}Y \to \mathfrak{O}X$ which maps every element U of $\mathfrak{O}Y$ to $h[U] = f^{-1}[U]$ (the inverse image of U under f) is a frame homomorphism.

1.1.26 Definition. A frame homomorphism $h\colon L \to M$ is **dense** if for every $x \in L$, $h(x) = 0$ implies that $x = 0$. This means that 0 is the only elment that h maps to 0. It is **codense** if for every $x \in L$, $h(x) = 1$ implies that $x = 1$, which also means that 1 is the only element that h maps to 1.

1.1.27 Definition. A frame homomorphism h is an **isomorphism** if it is one-one and onto. Two frames L and M are isomorphic (for which we write $L \cong M$) if there exists a frame isomorphism $g\colon L \to M$.

1.1.28 Property. *Any frame homomorphism $h\colon L \to M$ has a* **right adjoint** *$h_*\colon M \to L$ which preserves all meets and is not necessarily a frame homomorphism. It is defined by*
$$h_*(m) = \bigvee\{x \in L \mid h(x) \leq m\}, \forall m \in M.$$
The map h_* satisfies the following :

- *$id_L \leq h_*h$, where id_L is the identity map on L .*
- *$hh_* \leq id_M$.*
- *$hh_* = id_M$ if and only if h is onto.*
- *$h_*h = id_L$ if and only if h is one-one.*
- *If $g\colon M \to N$ is another frame homomorphism, then $(gh)_* = h_*g_*$.*
- *h is dense if and only if $h_*(0) = 0$.*

1.1.29 Example. *Let* **Top** *be the category of topological spaces and continuous maps and* **Frm** *the category of frames and frame homomorphisms, then there is a (contravariant) functor $\mathfrak{O}\colon$* **Top** $\to$ **Frm**, *mapping a topological space X to its frame of open sets $\mathfrak{O}X$ and mapping a continuous map $f\colon X \to Y$ to the map $\mathfrak{O}f\colon \mathfrak{O}Y \to \mathfrak{O}X$ given by $\mathfrak{O}f(U) = f^{-1}(U), \forall U \in \mathfrak{O}Y$. Since the map $\mathfrak{O}f$ is a frame homomorphism, then it has a right adjoint $f_*\colon \mathfrak{O}X \to \mathfrak{O}Y$ defined by $(\mathfrak{O}f)_*(U) = Y \setminus \overline{f(X \setminus U)}$ for every $U \in \mathfrak{O}X$.*

1.1.30 Remark. *The frame homomorphism $\bigvee\colon \mathfrak{D}L \to L$ which maps any downset U of L to its supremum $\bigvee U$ in L has a right adjoint defined by $\downarrow\colon L \to \mathfrak{D}L$ which maps an element $a \in L$ to $\downarrow a$.*

1.1.31 Definition. An **extension** of a frame L is a dense onto frame homomorphism $h\colon M \to L$. The extension h is **strict** if $h_*[L] = \{h_*(a) \mid a \in L\}$ generates M.

1.1.32 Definition. Let L be a frame and $a \in L$. The **pseudocomplement** of a is the largest element which misses a and is defined by $a^* = \bigvee\{x \in L \mid x \wedge a = 0\}$.

Let $a \in L$ and $\{a_\gamma \mid \gamma \in \Gamma\} \subseteq L$, then the following hold:

- $a \wedge a^* = 0$.
- $a \leq a^{**}$.
- $a^{***} = a^*$.
- $b^* \leq a^*$ whenever $a \leq b, \forall b \in L$.
- $(a \wedge b)^* = a^{**} \wedge b^{**}, \forall b \in L$.
- $(\bigvee\limits_{\gamma\in\Gamma} a_\gamma)^* = \bigwedge\limits_{\gamma\in\Gamma} a_\gamma^*$.
- if $h\colon L \to M$ is a dense onto frame homomorphism, then $h(a^*) = h(a)^*$.

1.1.33 Definition. An element $a \in L$ is **complemented** if $a \vee a^* = 1$. We denote by $C(L)$ the set of all complemented elements of the frame L. A **Boolean algebra** is a distributive lattice in which every element is complemented.

For a frame L, we define the set $\mathcal{B}L = \{a \in L \mid a = a^{**}\}$. The set $\mathcal{B}L$ is called the **Booleanization** of L as in [9] and it is a frame (Boolean algebra as well) with the following :

- $0_{\mathcal{B}L} = 0_L$.
- $1_{\mathcal{B}L} = 1_L$.
- the meet is the same as in L.
- the join is given by $\bigvee_{\mathcal{B}L} S = (\bigvee_L S)^{**}, \forall S \subseteq \mathcal{B}L$.

1.1.34 Remark. *If L is a Boolean algebra and $\{a_\gamma \mid \gamma \in \Gamma\} \subseteq L$, then $(\bigwedge\limits_{\gamma\in\Gamma} a_\gamma)^* = \bigvee\limits_{\gamma\in\Gamma} a_\gamma^*$.*

1.1.35 Remark.

- *For any bounded distributive lattice L, every proper filter $F \subseteq L$ is contained in an ultrafilter in L.*

- *Every ultrafilter in a frame is prime.*

1.1.36 Definition. A **Boolean filter** F in a frame L is a filter that is generated by its complemented elements. This means that $\forall x \in F, \exists N \subseteq F \cap C(L)$ such that $x = \bigvee N$.

1.1.37 Definition. A **maximal Boolean filter** in a frame L is a filter that is maximal in the sense of inclusion of Boolean filters in L.

1.1.38 Lemma. *[24] For a proper filter F in a Boolean algebra B, F is an ultrafilter in B if and only if F is prime, if and only if $\forall x \in B$, either $x \in F$ or $x^* \in F$.*

1.1.39 Theorem. *(Boolean Ultrafilter Theorem)[24] Every proper filter in a Boolean algebra B is contained in an ultrafilter in B.*

1.1.40 Lemma. *The mapping $b_L \colon L \to \mathcal{B}L$ given by $b_L(x) = x^{**}$, for every $x \in L$ is a dense onto frame homomorphism.*

1.1.41 Definition. An element a of a frame L is **dense** if $a^* = 0$.

1.1.42 Example. *For a topological space X, an open set $U \in \mathfrak{O}X$ is dense if and only if $\overline{U} = X$, which means that $U^* = X \setminus \overline{U} = \varnothing = 0_{\mathfrak{O}X}$.*

1.1.43 Definition. For any two elements a and b of a frame L, a is **rather below** b, written $a \prec b$, if there exists an element $s \in L$ such that $a \wedge s = 0$ and $b \vee s = 1$. This is equivalent to saying that $a^* \vee b = 1$.

1.1.44 Remark. *The* **rather below** *relation satisfies the following properties:*

- $0 \prec a \prec 1, \forall a \in L$.
- $a \leq b$ *whenever* $a \prec b, \forall a, b \in L$.
- $a \prec b, a \prec v$ *and* $u \prec b$ *whenever* $a \leq u \prec v \leq b, \forall a, b, u, v \in L$.
- $a \wedge u \prec b \wedge v$ *and* $a \vee u \prec b \vee v$ *whenever* $a \prec b$ *and* $u \prec v, \forall a, b, u, v \in L$.
- $b^* \prec a^*$ *whenever* $a \prec b, \forall a, b \in L$.
- $a^{**} \prec b$ *whenever* $a \prec b, \forall a, b \in L$.
- $h(a) \prec h(b)$ *whenever* $a \prec b, \forall a, b \in L$ *and for any frame homomorphism* $h \colon L \to M$.

1.1.45 Definition. A frame L is **regular** if $a = \bigvee\{x \in L \mid x \prec a\}, \forall a \in L$.

1.1.46 Property. *An element a in a frame L is* **complemented** *if and only if $a \prec a$, and then*

$$\{x \in L \mid x \prec a\} = \downarrow a.$$

1.1.47 Definition. For any two elements a and b in a frame L, a is **completely below** b, written $a \prec\!\!\prec b$, if there exists a sequence $\{x_r \mid r \in \mathbb{Q} \cap [0,1]\} \subseteq L$ satisfying:

- $x_0 = a,\ x_1 = b$.
- $x_p \prec x_q$ whenever $p < q, \forall p, q \in \mathbb{Q} \cap (0,1)$.

1.1.48 Remark. *The* **completely below** *relation satisfies the following properties:*

- $0 \prec\!\!\prec a \prec\!\!\prec 1, \forall a \in L$.
- $a \prec b$ *whenever* $a \prec\!\!\prec b, \forall a, b \in L$.
- $b^* \prec\!\!\prec a^*$ *whenever* $a \prec\!\!\prec b, \forall a, b \in L$.
- $a \prec\!\!\prec b$ *whenever* $a \leq u \prec\!\!\prec v \leq b, \forall a, b, u, v \in L$.
- $a^{**} \prec\!\!\prec b$ *whenever* $a \prec\!\!\prec b, \forall a, b \in L$.
- $a \wedge u \prec\!\!\prec b \wedge v$ *and* $a \vee u \prec\!\!\prec b \vee v$ *whenever* $a \prec\!\!\prec b$ *and* $u \prec\!\!\prec v, \forall a, b, u, v \in L$.
- *For a frame homomorphism* $h, h(a) \prec\!\!\prec h(b)$ *whenever* $a \prec\!\!\prec b, \forall a, b \in L$.
- $a \prec\!\!\prec b$ *implies that* $\exists c \in L$ *such that* $a \prec\!\!\prec c \prec\!\!\prec b, \forall a, b \in L$. *This property means that the* **completely below** *relation* **interpolates**.
- $a \prec\!\!\prec a$ *whenever* a *is a complemented element of* L.

1.1.49 Definition. A frame L is **competely regular** if $a = \bigvee\{x \in L \mid x \prec\!\!\prec a\}, \forall a \in L$.

1.1.50 Definition. A frame L is **normal** if

$$\forall x, y \in L \text{ such that } x \vee y = 1, \exists a, b \in L \text{ such that } a \wedge b = 0 \text{ and } x \vee a = y \vee b = 1.$$

1.1.51 Lemma. *In a normal frame* L, *the relations* $\prec\!\!\prec$ *and* $\prec$ *are the same. In other words,* $a \prec\!\!\prec b$ *if and only if* $a \prec b, \forall a, b \in L$.

1.1.52 Definition. A **point** in a frame L is a frame homomorphism $h \colon L \longrightarrow 2$.

For a frame L, we denote by ΣL the set of all points in L. For every $a \in L$, we define $\Sigma_a = \{h \colon L \longrightarrow 2 \mid h \text{ is a frame homomorphism and } h(a) = 1\}$.

Let $\tau = \{\Sigma_a \mid a \in L\}$, then the elements of τ satisfy the following:

1) $\Sigma_0 = \varnothing$.
2) $\Sigma_1 = \Sigma L$.
3) $\Sigma_a \cap \Sigma_b = \Sigma_{a \wedge b}, \forall a, b \in L$.
4) $\bigcup_{\gamma \in \Gamma} \Sigma_{a_\gamma} = \Sigma_{\bigvee a_\gamma}, \forall \{a_\gamma \mid \gamma \in \Gamma\} \subseteq L$.

Proof.

1) Since $h(0) = 0$ for every frame homomorphism, there is no frame homomorphism $h\colon L \to 2$ such that $h(0) = 1$ and then $\Sigma_0 = \varnothing$.

2) Since every frame homomorphism satisfies $h(1) = 1$ and all those defined from L to 2 in particular, then Σ_1 is the whole set of frame homomorphisms $h\colon L \to 2$ which is ΣL.

3) Let $a, b \in L$.

   - Let $h \in \Sigma_a \cap \Sigma_b$. Let us prove that $h \in \Sigma_{a\wedge b}$. We know that $h\colon L \to 2$ is a frame homomorphism such that $h(a) = 1$ and $h(b) = 1$, then

     $$h(a \wedge b) = h(a) \wedge h(b) = 1 \wedge 1 = 1.$$

     It follows that $h \in \Sigma_a \cap \Sigma_b$ and hence $\Sigma_a \cap \Sigma_b \subseteq \Sigma_{a\wedge b}$.
   - If $h \in \Sigma_{a\wedge b}$, then $h\colon L \to 2$ is a frame homomorphism such that

     $$1 = h(a \wedge b) = h(a) \wedge h(b).$$

     Furthermore, we have $1 = h(a) \wedge h(b) \leq h(a)$ and $1 = h(a) \wedge h(b) \leq h(b)$. This implies that $h(a) = 1$ and $h(b) = 1$ and then $h \in \Sigma_a \cap \Sigma_b$. Hence $\Sigma_{a\wedge b} \subseteq \Sigma_a \cap \Sigma_b$.

4) - If $h \in \bigcup_{\gamma\in\Gamma} \Sigma_{a_\gamma}$, then $\exists \gamma_0 \in \Gamma$ such that $h \in \Sigma_{a_{\gamma_0}}$, which means that $h(a_{\gamma_0}) = 1$. Also, $1 = h(a_{\gamma_0}) \leq \bigvee_{\gamma\in\Gamma} h(a_\gamma) = h(\bigvee_{\gamma\in\Gamma} a_\gamma)$, implying that $h(\bigvee_{\gamma\in\Gamma} a_\gamma) = 1$ and then $h \in \Sigma_{\bigvee_{\gamma\in\Gamma} a_\gamma}$. Hence $\bigcup_{\gamma\in\Gamma} \Sigma_{a_\gamma} \subseteq \Sigma_{\bigvee_{\gamma\in\Gamma} a_\gamma}$.
   - Assume that $h \in \Sigma_{\bigvee_{\gamma\in\Gamma} a_\gamma}$ and that, on the contrary, $h \notin \bigcup_{\gamma\in\Gamma} \Sigma_{a_\gamma}$. Then, $h\colon L \to 2$ is a frame homomorphism such that $h(\bigvee_{\gamma\in\Gamma} a_\gamma) = 1$. Also,

     $$\forall \gamma \in \Gamma, h(a_\gamma) \neq 1, \text{ which implies that } \forall \gamma \in \Gamma, h(a_\gamma) = 0$$

     because h is a frame homomorphism from L to 2. It follows that

     $$0 = \bigvee_{\gamma\in\Gamma} h(a_\gamma) = h(\bigvee_{\gamma\in\Gamma} a_\gamma).$$

     This is a contradicts our hypothesis and then $h \in \bigcup_{\gamma\in\Gamma} \Sigma_{a_\gamma}$. Hence

     $$\Sigma_{\bigvee_{\gamma\in\Gamma} a_\gamma} \subseteq \bigcup_{\gamma\in\Gamma} \Sigma_{a_\gamma}.$$

□

From the above properties of the elements of τ, it follows that τ is a topology on ΣL.

1.1.53 Remark. *The space* $(\Sigma L, \tau)$ *is a topological space called the* **spectrum** *of* L.

1.1.54 Lemma. *Let* $h\colon L \longrightarrow M$ *be a frame homomorphism and let* $\Sigma h\colon \Sigma M \longrightarrow \Sigma L$ *be defined by* $\Sigma h(\gamma) = \gamma \circ h$ *for every* γ *in* ΣM. *Then,* $(\Sigma h)^{-1}(\Sigma_a) = \Sigma_{h(a)}, \forall a \in L$, *which means that* Σh *is a continuous map.*

Proof. In fact, let $a \in L$.

- If $\gamma \in (\Sigma h)^{-1}(\Sigma_a)$, then $(\Sigma h)(\gamma) \in \Sigma_a$, which means that $1 = ((\Sigma h)(\gamma))(a) = (\gamma \circ h)(a)$. It follows that $\gamma \in \Sigma_{h(a)}$ and hence $(\Sigma h)^{-1}(\Sigma_a) \subseteq \Sigma_{h(a)}$.
- Conversely, if $\beta \in \Sigma_{h(a)}$, then $1 = \beta(h(a)) = (\beta \circ h)(a) = ((\Sigma h)(\beta))(a)$. This means that $(\Sigma h)(\beta) \in \Sigma_a$ and then $\beta \in (\Sigma h)^{-1}(\Sigma_a)$. Hence $\Sigma_{h(a)} \subseteq (\Sigma h)^{-1}(\Sigma_a)$.

□

1.1.55 Definition. An element x in a frame L is **meet-irreducible** or **prime** if it satisfies the following properties:

- $x \neq 1$.
- $a \wedge b \leq x$ implies that $a \leq x$ or $b \leq x, \forall a, b \in L$.

1.1.56 Definition. A topological space X is **sober** if every meet-irreducible in $\mathfrak{O}X$ is of the form $X \setminus \overline{\{x\}}$.

We can also define the spectrum in two other ways:

- Let $C_p(L)$ be the set of all completely prime filters in L.
  - For every element γ in ΣL, the set $F_\gamma = \{x \in L \mid \gamma(x) = 1\}$ is a completely prime filter.
  - On the other hand, for every completely prime filter $F \subseteq L$, the map $\gamma_F : L \to 2$, such that $\gamma_F(x) = 1$ if and only if $x \in F$, is in ΣL.

  In this case we can identify the space $(\Sigma L, \tau)$ with the space

  $$(\{F \subseteq L \mid F \text{ is completely prime }\}, \tau_1),$$

  where $\tau_1 = \{\Sigma_x \mid x \in L\}$ and $\Sigma_x = \{F \mid F \text{ is completely prime and } x \in F\}, \forall x \in L$.
- Let $mirr(L)$ be the set of all meet-irreducible (prime) elements in L.
  - Then $t_\gamma = \bigvee\{x \in L \mid \gamma(x) = 0\}$ is an element of $mirr(L), \forall \gamma \in \Sigma L$.
  - On the other hand, for any $x \in mirr(L)$, the element $\gamma_x\colon L \to 2$ defined for any $s \in L$ by $\gamma_x(s) = 1$ if and only if $x \nleq s$, is in ΣL.

Therefore, the space $(\Sigma L, \tau)$ can be identified with the space $(\{a \in L \mid a \in mirr(L)\}, \tau_2)$, where $\tau_2 = \{\Sigma_t \mid t \in L\}$ and $\Sigma_t = \{a \in mirr(L) \mid t \nleq a\}$.

1.1.57 Lemma. *[24, p.20] For any given frame L, the space ΣL is sober.*

1.1.58 Lemma. *[24, p.20] A topological space X is sober if and only if $\Sigma\mathfrak{O}X \cong X$.*

1.1.59 Remark.

- *Let $h\colon L \to M$ be a frame homomorphism and let p be a prime element in M, then $h_*(p)$ is a prime element in L.*
- *Every point $q \in \Sigma L$ induces (gives rise to) a completely prime filter $G_q = \{a \in L \mid a \nleq q\}$ and every completely prime filter $G \subseteq L$ induces a point $q_G = \bigvee(L \setminus G)$.*

1.1.60 Definition. [19] A frame L is **spatial** if $\forall a, b \in L$ such that $a \nleq b, \exists p \in \Sigma L$ such that $a \leq p$, but $b \nleq p$. This is equivalent to $a = \bigwedge\{p \in \Sigma L \mid a \leq p\}, \forall a \in L \setminus \{1\}$.

1.1.61 Definition. If L is a frame, we define the relation $\sqsubseteq$ as follows: for any two elements a and b in L, $a \sqsubseteq b$ if and only if $a \leq b$ and $a^* \nleq b$.

1.1.62 Definition. A frame L is **Hausdorff** if $a = \bigvee\{b \in L \mid b \sqsubseteq a\}, \forall a \in L \setminus \{1\}$.

1.1.63 Remark. *[18],[23] and [21] Let X be a topological space.*

- *X is T_0 if for every $a, b \in X$ such that $a \neq b$, either there exists an open set U such that $a \in U, b \notin U$ or an open set V such that $b \in V, a \notin V$.*
- *X is T_1 if for every $a, b \in X$ such that $a \neq b$, there exist open sets U and V such that $a \in U, b \notin U$ and $b \in V, a \notin V$. It is equivalent to saying that every singleton in X is closed.*
- *X is* **Hausdorff** *(or T_2) if for every $a, b \in X$ such that $a \neq b$, there exist open sets U and V such that $a \in U, b \in V$ and $U \cap V = \varnothing$. Every Hausdorff topological space is T_1 and sober([24, p.2]).*
- *If X is T_0, then X is Hausdorff if and only if $\mathfrak{O}X$ is a* **Hausdorff frame***.*

1.1.64 Definition. A frame L is T_1 if its points are the maximal elements in the poset $L \setminus \{1\}$. This means that $\forall p \in \Sigma L, \forall a \in L \setminus \{1\}, p \leq a$ implies that $a = p$.

1.1.65 Remark. *Every regular frame is Hausdorff and every Hausdorff frame is T_1.*

Covers and compactness

1.1.66 Definition. A **cover** of a frame L is a subset $C \subseteq L$ such that $\bigvee C = 1$. It is **countable** when it is of the form $C = \{c_n \mid n \in \mathbb{N}\}$.

A **subcover** of a cover C of a frame L is a subset $A \subseteq C$ such that $\bigvee A = 1$.

1.1.67 Definition. A frame L is **supercompact** if every cover of L contains 1_L.

1.1.68 Definition. A frame L is **compact** if every cover of L has a finite subcover. It is clear that every supercompact frame is compact.

1.1.69 Definition. An element s of a frame L is **compact** if for every $A \subseteq L$, $s \leq \bigvee A$ implies that there exists a finite $F \subseteq A$ such that $s \leq \bigvee F$.

1.1.70 Remark. *A frame L is compact if and only if 1_L is compact.*

1.1.71 Lemma. *Every frame which is compact and regular is normal.*

1.1.72 Definition. A cover U **refines** a cover V (or that U is a **refinement** of V) and is written $U \leq V$ if $\forall u \in U, \exists v \in V$ such that $u \leq v$.

Let $\mathcal{A}$ be a collection of covers of a frame L and $U, V \in \mathcal{A}$ and $a \in L$. We define the following sets:

- $Ua = \bigvee\{u \in U \mid u \wedge a \neq 0\}$.
- $U \wedge V = \{u \wedge v \mid u \in U, v \in V\}$.
- $UV = \{Uv \mid v \in V\}$.
- $U^{<\omega} = \{\bigvee S \mid S \subseteq_f U\}$, where $S \subseteq_f U$ means that S is a finite subset of U.

1.1.73 Remark.

- *If U is a cover of L and $V \subseteq L$ is such that $U \leq V$, then V is also a cover of L.*
- *The set $U^{<\omega}$ is a cover in L which is refined by U. In fact, $\{x\} \subseteq_f U, \forall x \in U$ so that $U \subseteq U^{<\omega}$, it implies that $1 = \bigvee U \leq \bigvee U^{<\omega}$, therefore $\bigvee U^{<\omega} = 1$. We also have $x \leq x = \bigvee\{x\} \in U^{<\omega}, \forall x \in U$ so that $U \leq U^{<\omega}$.*

1.1.74 Definition. Let $\mathcal{A}$ be a collection of covers in a frame L. An element $a \in L$ is **uniformly below** an element $b \in L$ relative to $\mathcal{A}$, written $a \lhd_{\mathcal{A}} b$, if $\exists U \in \mathcal{A}$ such that $Ua \leq b$.

The **uniformly below** relation satisfies the following properties:

- If $a \lhd_{\mathcal{A}} b$ and $a \lhd_{\mathcal{A}} c$, then $a \lhd_{\mathcal{A}} (b \wedge c), \forall a, b, c \in L$.
- If $a \lhd_{\mathcal{A}} c$ and $b \lhd_{\mathcal{A}} c$, then $(a \vee b) \lhd_{\mathcal{A}} c, \forall a, b, c \in L$.
- If $a \lhd_{\mathcal{A}} b$, then $a \prec b, \forall a, b \in L$.
- If $a \lhd_{\mathcal{A}} b$, then $a^{**} \lhd_{\mathcal{A}} b, \forall a, b \in L$.

1.1.75 Definition. A nonempty collection of covers $\mathcal{A}$ is **admissible** if

$$a = \bigvee\{b \in L \mid b \lhd_{\mathcal{A}} a\}, \forall a \in L.$$

1.1.76 Definition. A **uniformity** on a frame L is an admissible collection of covers $\mathcal{A}$ satisfying the following :

- If $U \in \mathcal{A}$ and V is a cover of L such that $U \leq V$, then $V \in \mathcal{A}$.
- If $U, V \in \mathcal{A}$, then $U \wedge V \in \mathcal{A}$.
- $\forall U \in \mathcal{A}, \exists V \in \mathcal{A}$ such that $VV \leq U$. The cover V is called a **star-refinement** of U.

If $\mathcal{A}$ satisfies only the first two properties, we say that $\mathcal{A}$ is a **nearness** on L.

1.1.77 Definition. A **uniform frame** $(L, \mathcal{A})$ is a frame L endowed with a uniformity $\mathcal{A}$. The covers in $\mathcal{A}$ will then be called **uniform covers** of L. If $\mathcal{A}$ is a nearness on L, we say that $(L, \mathcal{A})$ is a **nearness frame**.

Ring of continuous functions.

For a frame L, we set

$$\mathcal{R}L = \{h \colon \mathfrak{O}\mathbb{R} \to L \mid h \text{ is a frame homomorphism}\}$$

to be the ring of all real-valued continuous functions on a frame L, and we define the **cozero map** coz: $\mathcal{R}L \to L$ by: $\forall \alpha \in \mathcal{R}L, \text{coz}\alpha = \alpha(\mathbb{R} \setminus \{0\})$. An element $s \in L$ is a **cozero element** if $s = \text{coz}\alpha$ for some $\alpha \in \mathcal{R}L$. We denote by coz$L$ the set of all the cozero elements of L and it is called the **cozero part** of L. For $s, t \in L$ such that $s \prec\!\!\prec t, \exists u \in \text{coz}L$ such that $s \prec\!\!\prec u \prec\!\!\prec t$. Also, there is a cozero element b such that $a \wedge b = 0$ and $b \vee t = 0$.

1.2 Sublocales and localic maps

In this section we study sublocales and some of their properties.

Let **Loc** denote the dual category of **Frm**, this means that objects in **Loc** are frames but the arrows in **Loc** are opposite from those in **Frm**. If $h \colon L \to M$ is a frame homomorphism, then $h_* \colon M \to L$ is in **Loc**. Objects in **Loc** are called **locales** and arrows in **Loc** are called **localic maps**.

1.2.1 Definition. [24] A **localic map** $t \colon M \to L$ is a map between frames which preserves all meets and has its left adjoint $t^* \colon L \to M$ preserving finite meets.

1.2.2 Definition. A **sublocale** of a locale L is a nonempty subset $S \subseteq L$ such that the inclusion map $j \colon S \hookrightarrow L$ is a localic map.

1.2.3 Example. *Let L be a frame and $a \in L$. Then the set $\uparrow(a) = \{x \in L \mid x \geq a\}$ is a sublocale of L.*

1.2.4 Definition. Let L be a frame. The **Heyting implication** on L is the binary operation $\to$ defined by: $x \to y = \bigvee\{a \in L \mid x \wedge a \leq y\}, \forall x, y \in L$.

1.2.5 Proposition. *Any sublocale S of a frame L satisfies the following properties:*

- $\bigwedge X \in S, \forall X \subseteq S$. *This property implies that $1 = \bigwedge \varnothing$ belongs to every sublocale $S \subseteq L$.*
- $x \to s \in S, \forall s \in S$ *and* $\forall x \in L$.

In all our work, $\mathcal{S}\ell(L)$ represents the collection of all the sublocales of a locale L. Two sublocales S and T in $\mathcal{S}\ell(L)$ meet when $S \cap T \neq \mathsf{O}$. Where the set $\mathsf{O} = \{1_L\}$ is the **least sublocale** of L.

1.2.6 Remark. *The collection $\mathcal{S}\ell(L)$ of all the sublocales of a frame L is a lattice with the following properties:*

- $0_{\mathcal{S}\ell(L)} = \mathsf{O}$.
- $1_{\mathcal{S}\ell(L)} = L$.
- *The meet operation is given by the intersection.*
- *The join operation is defined by*

$$\bigvee_{\gamma\in\Gamma} T_\gamma = \{\bigwedge T \mid T \subseteq \bigcup_{\gamma\in\Gamma} T_\gamma\}, \forall\{T_\gamma \mid \gamma \in \Gamma\} \subseteq \mathcal{S}\ell(L).$$

1.2.7 Definition. The **closure** of a sublocale T in a frame L is the smallest closed sublocale containing T and it is given by $\overline{T} = \uparrow(\bigvee T) = \uparrow(\mathfrak{c}(\bigvee T))$. If S and T are sublocales in L such that $S \subseteq T$, then the closure of S in T is given by $\overline{S}^T = \overline{S} \cap T$.

For any two sublocales S and T of a frame L, the following hold :

- $\overline{\mathsf{O}} = \mathsf{O}$.
- $S \subseteq T$ implies that $\overline{S} \subseteq \overline{T}$.
- $\overline{\overline{T}} = T$.
- $\overline{S \cap T} \subseteq \overline{T} \cap \overline{S}$.

1.2.8 Definition. A sublocale T is **dense** in a locale L if $\overline{S} = L$.

1.2.9 Lemma. *[24, p.40] The Booleanization of L is the* **least dense sublocale** *of L.*

1.2.10 Example. *If $h\colon M \to L$ is an extension of a frame L, then $h_*[L]$ is a dense sublocale of M.*

1.2.11 Proposition. *Let s be an element of a frame L, we denote by*

$$\mathfrak{c}(s) = \{x \in L \mid a \leq x\} = \uparrow s,$$

the **closed sublocale associated with** *s and we write*

$$\mathfrak{o}(s) = \{s \to a \mid a \in L\},$$

the **open sublocale associated with** *s. Let $a, b \in L$ and $\{a_\lambda \mid \lambda \in \Lambda\} \subseteq L$, then we have the following properties:*

- *$\mathfrak{o}(a)$ and $\mathfrak{c}(a)$ are* **complements** *in $\mathcal{S}\ell(L)$. This means that $\mathfrak{o}(a) \cap \mathfrak{c}(a) = O$ and $\mathfrak{o}(a) \vee \mathfrak{c}(a) = L$. This is also equivalent to saying that $a \leq b$ if and only if $\mathfrak{c}(b) \subseteq \mathfrak{c}(a)$, if and only if $\mathfrak{o}(a) \subseteq \mathfrak{o}(b)$.*
- $\mathfrak{o}(a) \cap \mathfrak{o}(b) = \mathfrak{o}(a \wedge b)$.
- $\mathfrak{c}(a) \vee \mathfrak{c}(b) = \mathfrak{c}(a \wedge b)$.
- $\bigvee\limits_{\lambda \in \Lambda} \mathfrak{o}(a_\lambda) = \mathfrak{o}(\bigvee\limits_{\lambda \in \Lambda} a_\lambda)$.
- $\bigcap\limits_{\lambda \in \Lambda} \mathfrak{c}(a_\lambda) = \mathfrak{c}(\bigvee\limits_{\lambda \in \Lambda} a_\lambda)$.
- *$\mathfrak{o}(a) \cap S \neq O$ if and only if $\mathfrak{o}(a) \cap \overline{S} \neq O$.*
- $\overline{\mathfrak{c}(a)} = \mathfrak{c}(a)$.
- $\overline{\mathfrak{o}(a)} = \mathfrak{c}(a^*) = \uparrow a^*$.

1.2.12 Definition. For a frame L, a **nucleus** is a map $\nu \colon L \to L$ satisfying the following $\forall x, y \in L$:

1) $x \leq \nu(x)$.
2) $\nu(x) \leq \nu(y)$ whenever $x \leq y$.
3) $\nu\nu(x) = \nu(x)$.
4) $\nu(x \wedge y) = \nu(x) \wedge \nu(y)$.

1.2.13 Remark. *[24] The following properties hold for any given frame L:*

- *Let $S \subseteq L$ be a* **sublocale** *of a frame L, then the localic map (inclusion map) $j_S \colon S \hookrightarrow L$ is given and we can define the* **nucleus associated with** *S by :*

$$\nu_S(a) = j_S^*(a) = \bigwedge\{x \in S \mid a \leq x\}.$$

- *For a* **nucleus** *$\nu \colon L \to L$, the S_ν* **sublocale associated with** *ν is defined by $S_\nu = \nu[L]$.*

- *Let $h\colon L \to M$ be an* **onto frame homomorphism** *(viewed as a sublocale homomorphism), then the* **sublocale associated with** *h is defined by $S_h = h_*[L]$.*

- *For a* **nucleus** *$\nu\colon L \to L$, the* **localic map** *$h_\nu\colon L \to \nu[L]$* **associated with** *ν is defined by $h_\nu(a) = \nu(a), \forall a \in L$.*

- *The* **nucleus associated with a localic map** *$h\colon L \to M$ is defined by $\nu_h = h_* h$.*

1.2.14 Proposition. *[24] Let $S \subseteq L$ be a sublocale of a frame L and ν_S the associated nucleus.*

- *The closed sublocales in S are exactly the elements of the set $\{\mathfrak{c}_S(s) \mid s \in L\}$, where $\mathfrak{c}_S(s) = \mathfrak{c}(s) \cap S = \mathfrak{c}(\nu_S(s)), \forall s \in L$.*

- *The open sublocales in S are exactly the elements of the set $\{\mathfrak{o}_S(s) \mid s \in L\}$, where $\mathfrak{o}_S(s) = \mathfrak{o}(s) \cap S = \mathfrak{o}(\nu_S(s)), \forall s \in L$.*

In all our work, Sub(L) represents the collection of all complemented locales in $\mathcal{S}\ell(L)$, and $\mathcal{O}(L)$ represents the collection of all open sulocales in $\mathcal{S}\ell(L)$.

1.2.15 Lemma. *[19] Every nontrivial compact locale has at least one point.*

1.2.16 Lemma. *[24, p.104] If $S \subseteq L$ is a sublocale of a frame L, then $\Sigma S = \Sigma L \cap S$.*

1.3 Stone-Čech compactification

1.3.1 Definition. [2] A **compactification** of a frame L is the pair (M, h), where M is a compact regular frame and $h\colon M \to L$ is a dense onto frame homomorphism.

1.3.2 Definition. An **ideal** I in a frame L is a nonempty subset $I \subseteq L$ satisfying the following properties:

- $0 \in I$.

- $a \vee b \in I, \forall a, b \in I$.

- $b \leq a$ implies that $b \in I, \forall a \in I, \forall b \in L$.

We set $\mathfrak{J}L$ as the frame of all ideals in a frame L. We say that an ideal I is **proper** if it is different from L.

1.3.3 Definition. An ideal $I \in \mathfrak{J}L$ is **completely regular** if $\forall a \in I, \exists b \in I$ such that $a \prec\!\!\prec b$.

1.3.4 Example. *For any $a \in L$, the set $r_L(a) = \{x \in L \mid x \prec\!\!\prec a\}$ is a completely regular ideal in L.*

We set $\mathcal{CR}\mathfrak{J}L$ as the subset of $\mathfrak{J}L$ made up of all completely regular ideals in L.

1.3.5 Lemma. *[24] $\mathcal{CR}\mathfrak{J}L$ is a regular frame with the following:*

- $0_{\mathcal{CR}\mathfrak{J}L} = 0_{\mathfrak{J}L} = \{0\}$.
- $1_{\mathcal{CR}\mathfrak{J}L} = 1_{\mathfrak{J}L} = L$.
- *The meet is given by intersection.*
- *The join is given by*

$$\bigvee_{\mathcal{CR}\mathfrak{J}L} \{I_\gamma \mid \gamma \in \Gamma\} = \{S \mid S \text{ is finite and } S \subseteq \bigvee_{\gamma\in\Gamma} I_\gamma\}, \forall \{I_\gamma \mid \gamma \in \Gamma\} \subseteq \mathcal{CR}\mathfrak{J}L.$$

1.3.6 Remark. *The join of a collection S of ideals is not necessarily an ideal, unless if S is updirected.*

1.3.7 Definition. The **Stone-Čech compactification** of a frame L is the pair $(\beta L, j_L)$ such that $\beta L = \mathcal{CR}\mathfrak{J}L$ and $j_L = \bigvee \colon \beta L \to L$ maps a completely regular ideal $I \subseteq L$ to its join $\bigvee I$.

1.3.8 Remark. *Since the map j_L is a frame homomorphism, it has a right adjoint $r_L \colon L \to \beta L$ defined by $r_L(a) = \{x \in L \mid x \prec\!\!\prec a\}, \forall a \in L$. In all our work, we will often write r_L as r for a frame L. The map r_L satisfies the following properties:*

1) $r_L(a^*) = r_L(a)^*, \forall a \in L$.
2) $I = \bigvee_{s\in I} r_L(s), \forall I \in \beta L$.

1.3.9 Lemma. *If $h \colon M \to L$ is a* **compactification** *of a frame L, then there is a unique map $i \colon M \to \beta L$ such that $h = \bigvee i$, where $i(x) = \bigcup\{\downarrow h(z) \mid z \prec x\}$ for every $x \in M$.*

1.3.10 Theorem. *For every frame homomorpism $h \colon L \to M$, there exists a frame homomorphism $\bar{h} \colon \beta L \to \beta M$ such that the diagram below is commutative. In other words, $\bigvee \bar{h}(I) = h \bigvee(I), \forall I \in \beta L$.*

$$\begin{tikzcd}
\beta L \arrow[r, "\bar{h}"] \arrow[d, "\bigvee = j_L"'] & \beta M \arrow[d, "j_M = \bigvee"] \\
L \arrow[r, "h"'] & M
\end{tikzcd}$$

2. Convergence in frames

Firstly, we describe the pointfree equivalent of concepts of convergence and clustering of classical filters by using covers. We then construct the strict extension associated to a given collection of filters in a frame as in [6]. Then we try to mimic this previous convergence in terms of more general types of filters than usual ones. This chapter is mostly based on the work in [17], [11], [7], [22] and [14]. Concerning general knowledge about topology and category theory we refer to [21] and [1] respectively.

2.1 Convergence of classical filters

Convergence of filters in a frame.

2.1.1 Definition. [17] A filter F in a frame L is **convergent** (or **converges**) if it meets every cover of L.

2.1.2 Definition. [8] A filter F in a frame L is **strongly convergent** (or **strongly converges**) if there is a completely prime filter F_p such that $F_p \subseteq F$.

2.1.3 Property. *By definition, every completely prime filter is strongly convergent.*

2.1.4 Property. *Every completely prime filter is convergent.*

In fact if F_p is a completely prime filter and S is a cover in L, we have $\bigvee S = 1 \in F_p$ and this implies that $F_p \cap S \neq \emptyset$, then F_p is convergent.

2.1.5 Property. *A filter F in a frame L which contains a convergent filter F_c is convergent.*

In fact for every cover S in L, we have $\varnothing \neq F \cap S \subseteq F_c \cap S$.

2.1.6 Property. *Every strongly convergent filter in a frame L is convergent.*

In fact if F is strongly convergent, then it contains a completely prime filter and then F converges by the previous property. The converse is not true in general.

2.1.7 Definition. A filter F in a frame L is **clustered** (or **clusters**) if sec$F$ meets every cover of L.

2.1.8 Definition. A filter F in a frame L is **strongly clustered** (or **strongly clusters**) if $\exists p \in \Sigma L$ such that $\bigvee\{x^* \mid x \in F\} \leq p$. This is equivalent to saying that F is contained in a strongly convergent filter in L.

For a frame L and a filter $F \subseteq L$, we define sec$F = \{x \in L \mid \forall a \in F, a \wedge x \neq 0\}$.

2.1.9 Remark. *Every filter F which strongly clusters is clustered.*

In fact, if F strongly clusters, $\exists p \in \Sigma L$ such that $\bigvee\{x^* \mid x \in F\} \leq p$. Therefore,

$$\bigvee\{x^* \mid x \in F\} \leq p \neq 1$$

because $p \in \Sigma L$. This implies that $\bigvee\{x^* \mid x \in F\} \neq 1$, hence F clusters. But as the example below shows, the converse does not always hold.

2.1.10 Lemma. *[24]* $\Sigma\mathcal{B}\mathfrak{O}\mathbb{R} = \varnothing$.

Proof. In fact, suppose that $\Sigma\mathcal{B}\mathfrak{O}\mathbb{R} \neq \varnothing$. Then, $\exists U \in \mathfrak{O}\mathbb{R}$ such that $U^* = \mathbb{R} \setminus \overline{U}$ is meet-irreducible in $\mathcal{B}\mathfrak{O}\mathbb{R}$. By Lemma(1.2.9), $\mathcal{B}\mathfrak{O}\mathbb{R}$ is a sublocale of $\mathfrak{O}\mathbb{R}$ and then by Lemma(1.2.16), $\Sigma\mathcal{B}\mathfrak{O}\mathbb{R} \subseteq \Sigma\mathfrak{O}\mathbb{R}$. It follows that U^* is also a meet-irreducible in $\mathfrak{O}\mathbb{R}$. Also, by Lemma (1.1.57), combined with the Hausdorffness and then the sobriety (every Hausdorff space is sober as shown in [24, p.2]) of $\mathbb{R}$, then $\Sigma\mathfrak{O}\mathbb{R} \cong \mathbb{R}$. This means that the meet-irreducibles in $\Sigma\mathfrak{O}\mathbb{R}$ can be identified with those in $\mathbb{R}$ and they are of the form $\mathbb{R}\setminus\overline{\{x\}}, x \in \mathbb{R}$. Finally, we have $U^* = \mathbb{R}\setminus\overline{U} = \mathbb{R}\setminus\overline{\{x_0\}}$, for some $x_0 \in \mathbb{R}$. This is possible if and only if $U = \{x_0\}$, meaning that $\{x_0\}$ is open in $\mathbb{R}$ and this contradicts the fact that $\mathbb{R}$ is a T_1-space (Hausdorff space) because in a T_1-space, every singleton is a closed set. Hence $\Sigma\mathcal{B}\mathfrak{O}\mathbb{R} = \varnothing$.

□

2.1.11 Example. *Let us consider the frame $L = \mathcal{B}\mathfrak{O}\mathbb{R}$. Therefore this frame cannot contain any completely prime filter because if it has one, say F by Remark (1.1.59), there is a correspondent point $p_F \in \Sigma L$, which contradicts the fact that $\Sigma\mathcal{B}\mathfrak{O}\mathbb{R} = \varnothing$. This implies that L does not contain any strongly convergent filter because if it contained one, this convergent filter would contain a completely prime one. If $0 \neq a \in L$, then the filter $\uparrow a$ is clustered. In fact,*

$$\bigvee\{x^* \mid x \in\uparrow a\} = \bigvee\{x^* \mid x \geq a\} \leq \bigvee\{x^* \mid x^* \leq a^*\} = a^* \neq 1 \text{ because } a \neq 0.$$

This means the filter $\uparrow a$ clusters but does not strongly clusters because it is not contained in any strongly convergent filter.

Some properties of filters.

2.1.12 Property. *Every convergent filter in a frame L clusters. In fact, if F is a convergent filter in L, for every cover S of L, $\varnothing \neq F \cap S \subseteq (\text{sec}F) \cap S$.*

2.1.13 Property. *A maximal filter in a frame is convergent if and only if it is clustered.*

In fact,

- If $1 \wedge x = x \neq 0, \forall x \in F$, then $1 \in \text{sec}F$ so that $\text{sec}F \neq \varnothing$.
- $0 \notin \text{sec}F$ because $0 \wedge x = 0, \forall x \in F$.

- If $x, y \in \text{sec}F$, assume that $a \in F$, therefore $x \wedge a \neq 0$ and $y \wedge a \neq 0$ and then

$$(x \wedge y) \wedge a = (x \wedge a) \wedge (y \wedge a) \neq 0,$$

  so that $x \wedge y \in \text{sec}F$.

- If $x \in \text{sec}F$ and $y \in L$ such that $x \leq y$, then if $a \in F$, $x \wedge a \leq y \wedge a$, since $x \wedge a \neq 0$, then $y \wedge a \neq 0$. It follows that $y \in \text{sec}F$.

Therefore sec$F$ is a filter containing F and then sec$F = F$ by maximality. It follows that if S is any cover of L, then $F \cap S = (\text{sec}F) \cap S \neq \varnothing$.

2.1.14 Proposition. *[5] In a regular frame L, a filter is convergent if and only if it is strongly convergent.*

Proof.

- Let L be a regular frame. Let F be a filter in L which contains a completely prime filter F_p . Let S be a cover of L. To see that $S \cap F \neq \varnothing$, we have $1 = \bigvee S \in F_p$ and since F_p is completely prime, we must have F_p meets S.

- As for the converse, let F be a convergent filter in L. We set

$$\text{reg}F = \{t \in L \mid x \prec t \ \text{ for some } \ x \in F\}.$$

  Let us prove that reg$F$ is a completely prime filter contained in F.

  - Obviously, $F \ni 1 \prec 1$ so that reg$F \neq \varnothing$.
  - Let t be an element of reg$F$, then $t \in L$ and $x \prec t$ for some $x \in F$, since $x \prec t$ implies that $x \leq t$, then $t \in F$ because F is a filter. This proves that reg$F \subseteq F$.
  - Since $0 \notin F$, then $0 \notin \text{reg}F$.
  - If $s, t \in \text{reg}F$, then $\exists x, y \in F$ such that $x \prec s$ and $y \prec t$, this implies that $x \wedge y \in F$ and $x \wedge y \prec s \wedge t$. It follows that $s \wedge t \in \text{reg}F$.
  - Let M be a subset of L such that $\bigvee M \in \text{reg}F$, then there exists $x \in F$ such that $x \prec \bigvee M$, this means that

$$x^* \vee (\bigvee M) = \bigvee (M \cup \{x^*\}) = 1.$$

    Hence $M \cup \{x^*\}$ is a cover of L and then meets F. Let us consider the set

$$S = \{a \in L \mid a \prec t \text{ for some } t \in M\}.$$

    Let us prove that $S \cup \{x^*\}$ is a cover of L. Let $t \in M$, then by regularity,

$$t = \bigvee \{x \in L \mid x \prec t\}.$$

Since $\{x \in L \mid x \prec t\} \subseteq S$, then $t = \bigvee\{x \in L \mid x \prec t\} \leq \bigvee S$ and it follows that $t \leq \bigvee S, \forall t \in M$. Hence $\bigvee M \leq \bigvee S$, which implies that

$$1 = \bigvee(M \cup \{x^*\}) = x^* \vee (\bigvee M) \leq x^* \vee (\bigvee S) = \bigvee(S \cup \{x^*\}).$$

Therefore $\bigvee(S \cup \{x^*\}) = 1$ and hence $S \cup \{x^*\}$ is a cover of L. Then $F \cap S \neq \varnothing$ and it means that $\exists b \in F, \exists t \in M$ such that $b \prec t$, which implies that $b \leq t$. It follows that $b \in F$ because F is a filter containing t, then $F \cap M \neq \varnothing$. Hence the filter reg$F$ is completely prime.

Hence, F is strongly convergent because it contains the completely prime filter reg$F$. □

2.1.15 Theorem. *[17] Let L be a frame. A filter F in L is clustered if and only if*

$$\bigvee\{x^* \mid x \in F\} \neq 1.$$

Proof.

- Assume that L is clustered and $\bigvee\{x^* \mid x \in F\} = 1$. Then $\{x^* \mid x \in F\}$ is a cover of L, therefore $\exists y \in F$ such that $y^* \in$ sec$F$. This means that $\forall t \in F, y \wedge t \neq 0$, but we have $y \wedge y^*$ and $y \in F$, which contradicts the hypothesis. Hence, $\bigvee\{x^* \mid x \in F\} \neq 1$.
- Now assume that $\bigvee\{x^* \mid x \in F\} \neq 1$ and F is not clustered. Then there is a cover C of L such that $C \cap (\text{sec}F) = \varnothing$. So for any $t \in C, t \notin$ sec$F$ and then $\exists x_t \in F$ such that $t \wedge x_t = 0$, which implies that $t \leq x_t{}^*$. This means that $\forall t \in C, \exists x_t \in \{x^* \mid x \in F\}$ so that C refines the set $\{x^* \mid x \in F\}$ which is a cover of L by Remark (1.1.73). This contradicticts the hypothesis. Therefore F clusters.

□

2.1.16 Definition. [17] A frame L is **almost compact** if every cover of L contains a finite subset S such that $(\bigvee S)^* = 0$.

2.1.17 Lemma. *[23] A frame L is almost compact if and only if for any filter F in L,*

$$\bigvee\{x^* \mid x \in F\} \neq 1.$$

Proof.

- If L is almost compact, assume that there is a filter $F \subseteq L$ such that $\bigvee\{x^* \mid x \in F\} = 1$. Therefore the set $\{x^* \mid x \in F\}$ is a cover and by hypothesis, $\exists p \in \mathbb{N}$ such that $x_1, x_2, \ldots, x_p \in F$ and $0 = (x_1^* \vee x_2^* \vee \ldots \vee x_p^*)^* = x_1^{**} \wedge x_2^{**} \wedge \ldots \wedge x_p^{**}$. On the other hand, since $x_1, x_2, \ldots, x_p \in F$ and $x_i \leq x_i^{**}$ for $i = 1, 2, \ldots p$, this implies that $x_i^{**} \in F$ for $i = 1, 2, \ldots p$. Therefore $x_1^{**} \wedge x_2^{**} \wedge \ldots \wedge x_p^{**} \in F$ because F is a filter. It follows that $0 = x_1^{**} \wedge x_2^{**} \wedge \ldots \wedge x_p^{**} \in F$, which contradicts the fact that $0 \notin F$. Hence, for any filter F in L, $\bigvee\{x^* \mid x \in F\} \neq 1$.

- If for any filter F in L, $\bigvee\{x^* \mid x \in F\} \neq 1$, assume that L is not almost compact. Then, there is a cover C of L such that $(\bigvee K)^* \neq 0$, for every finite subset $K \subseteq C$. We set $G = \{x \in L \mid x \geq (\bigvee K)^* \text{ for some finite } K \subseteq C\}$. Let us prove that G is a filter in L.
  - If $0 \in G$, then $(\bigvee K)^* = 0$ for some finite $K \subseteq C$ and this contradicts the hypothesis, hence $0 \notin G$.
  - If $L \ni a \geq b \in G$, then $a \geq b \geq (\bigvee K)^*$ for some finite $K \subseteq C$ and hence $a \in G$.
  - If $a, b \in G, \exists m, n \in \mathbb{N}$ such that $a = (a_1 \vee a_2 \vee \ldots \vee a_n)^*$ and $b = (b_1 \vee b_2 \vee \ldots \vee b_m)^*$, where $a_i, b_j \in C$ for $i = 1, 2, \ldots, n$ and for $j = 1, 2, \ldots, m$. Therefore

    $$a \wedge b = (a_1 \vee a_2 \vee \ldots \vee a_n)^* \wedge (b_1 \vee b_2 \vee \ldots \vee b_m)^* = (a_1 \vee a_2 \vee \ldots \vee a_n \vee b_1 \vee b_2 \vee \ldots \vee b_m)^*.$$

    Hence, $a \wedge b \in G$.

  Therefore, G is a filter in L and we get:

  $$1 = \bigvee\{c \mid c \in C\} \leq \bigvee\{c^{**} \mid c \in C\} \leq \bigvee\{x^* \mid x \in G\}.$$

  This is because $\forall c \in C, c^* \geq c^* = (\bigvee\{c\})^*$, therefore $c^* \in G$ and then

  $$\{c^{**} \mid c \in C\} \subseteq \{x^* \mid x \in G\}.$$

  We finally have $\bigvee\{x^* \mid x \in G\} = 1$, which contradicts our hypothesis on all filters in L. Hence, L is almost compact.

□

2.1.18 Lemma. *[23] A regular frame L is compact if and only if it is almost compact.*

Proof.

- Assume that L is compact and let C be a cover of L. Then, $\exists D \subseteq C$ such that D is a finite subcover of C, this means that $\bigvee D = 1$, therefore $(\bigvee D)^* = 0$. This proves that L is almost compact.

- Assume that L is almost compact and let $C = \{x_\alpha \mid \alpha \in \Lambda\}$ be a cover of L. By regularity,

$$\begin{aligned}
x_1 &= \bigvee\{y_{j_1} \in L \mid y_{j_1} \prec x_1 \text{ and } j_1 \in \Gamma_1\} \\
x_2 &= \bigvee\{y_{j_2} \in L \mid y_{j_2} \prec x_2 \text{ and } j_2 \in \Gamma_2\} \\
&\vdots \\
x_\beta &= \bigvee\{y_{j_\beta} \in L \mid y_{j_\beta} \prec x_\beta \text{ and } j_\beta \in \Gamma_\beta \text{ and } \beta \in \Lambda\} \\
&\vdots
\end{aligned}$$

where the Γ_α's are disjoint. Therefore the join over the x_α's is the same as the join over the y_{j_α}'s. Then,

$$1 = \bigvee C = \bigvee\{y_{j_\alpha} \in L \mid y_{j_\alpha} \prec x_\alpha \text{ and } j_\alpha \in \Gamma_\alpha \text{ and } \alpha \in \Lambda\},$$

which will therefore form a cover. By almost compactness, there is a finite

$$S = \{s_1, s_2, \ldots, s_p\} \subseteq (\Gamma_1 \cup \Gamma_2 \cup \ldots \Gamma_\alpha \ldots)$$

such that $(\bigvee\{y_s \mid s \in S\})^* = 0$ and which means that $(\bigvee\{y_s \mid s \in S\})^{**} = 1$.
Now let us prove that $(\bigvee\{y_s \mid s \in S\})^{**} \prec \bigvee\{x_\alpha \mid \alpha \in \Lambda$ and $s \in \Gamma_\alpha$ for $s \in S\}$. In fact, If $s \in S$, then $y_s \prec x_\alpha, \forall\alpha$ such that $\alpha \in \Lambda,$ and $s \in \Gamma_\alpha$. Therefore,

$$\bigvee\{y_s \mid s \in S\} \prec \bigvee\{x_\alpha \mid \alpha \in \Lambda \text{ and } s \in \Gamma_\alpha \text{ for } s \in S\},$$

which implies that

$$\underbrace{(\bigvee\{y_s \mid s \in S\})^{**}}_{1} \prec \bigvee\{x_\alpha \mid \alpha \in \Lambda \text{ and } s \in \Gamma_\alpha \text{ for } s \in S\}.$$

Finally, we get that $\bigvee\{x_\alpha \mid \alpha \in \Lambda$ and $s \in \Gamma_\alpha$ for $s \in S\} = 1$. Since S is finite and the Γ_α' s are disjoint, we get a finite number of such α's in Λ, this means that the set $\{x_\alpha \mid \alpha \in \Lambda$ and $s \in \Gamma_\alpha$ for $s \in S\}$ is a finite subcover of C. Hence L is compact.

□

2.1.19 Remark.

a) *A filter contained in a strongly clustered filter strongly clusters. In fact, let G and F be two filters in a frame L such that $G \subseteq F$ and F strongly clusters in L. This means that $\exists p \in \Sigma L$ such that $\bigvee\{a^* \mid a \in F\} \leq p$. Since $\{a^* \mid a \in G\} \subseteq \{a^* \mid a \in F\}$, we get $\bigvee\{a^* \mid a \in G\} \leq \bigvee\{a^* \mid a \in F\} \leq p$. Hence, G strongly clusters.*

b) *A filter which converges in a regular frame strongly clusters. In fact, if L is a regular frame and G is a convergent filter in L, then G is strongly convergent which means that it contains a completely prime filter say F and by Remark (1.1.59) there is an element $p_F = \bigvee(L \setminus F) \in \Sigma L$. Since $F \subseteq G$, we get that $L \setminus G \subseteq L \setminus F$. Therefore $\forall b \in G, b^* \in L \setminus G \subseteq L \setminus F$ (because G is a filter). This means that $\forall x \in G, x^* \in L \setminus F$. On the other hand, $F = \{a \in L \mid a \not\leq t\}$ and it implies that $b^* \leq t, \forall b \in G$, therefore $\bigvee\{b^* \mid b \in G\} \leq t$ and hence, G strongly clusters.*

2.1.20 Proposition. *In a regular frame L, a filter strongly clusters if and only if it is contained in a convergent one.*

Proof. Let L be a regular frame.

- Suppose that F strongly clusters in L, then $\exists q \in \Sigma L$ such that $\bigvee\{a^* \mid a \in F\} \leq q$, so by Remark (1.1.59) the set $F_q = \{x \in L \mid x \nleq q\}$ is a completely prime filter in L. Let us prove that the set $M = \{a \wedge b \mid a \in F \text{ and } b \in F_q\}$ is a base for a proper filter $G \subseteq L$. In fact we have the following:

  - If $0 \in M$, then $\exists a \in F$ and $b \in F_q$ such that $a \wedge b = 0$, so that

  $$b \leq a^* \leq \bigvee\{a^* \mid a \in F\} \leq q.$$

  This is impossible because the fact that $b \in F_q$ implies that $b \nleq q$. Hence, $0 \notin M$.

  - If $m_1, m_2 \in M, m_1 = a_1 \wedge b_1, m_2 = a_2 \wedge b_2$ for some $a_1, a_2 \in F$ and $b_1, b_2 \in F_q$, then

  $$m_1 \wedge m_2 = (a_1 \wedge b_1) \wedge (a_2 \wedge b_2) = \underbrace{(a_1 \wedge a_2)}_{\in F} \wedge \underbrace{(b_1 \wedge b_2)}_{\in F_q}.$$

  This means that $m_1 \wedge m_2 \in M$. Hence M is closed under finite meets.

  Since $1 \in F$ and $1 \in F_q$, then $\forall x \in F, x \wedge 1 = x \in M$ so that $F \subseteq M \subseteq G$ and $\forall y \in F_q, 1 \wedge y = y \in M$ so that $F_q \subseteq M \subseteq G$. Since F_q is a completely prime filter, $F_q \subseteq G$ means that G converges. Hence G is a convergent filter containing F.

- Let F and G be filters in L such that $F \subseteq G$ and G converges in L. Then by the second part of Remark (2.1.19), G strongly clusters and then F strongly clusters by the first part of the same remark.

$\square$

2.1.21 Remark. *Let $h : L \to M$ be a frame homomorphism, we have:*

1) For any filter F in M, $h^{-1}[F]$ is also a filter in L. In fact:

  - *Assume that $0 \in h^{-1}[F]$, this means that $\exists x \in F$ such that $h^{-1}(x) = 0$, so that $x = h(0) = 0$, which contradicts the fact that $0 \neq x \in F$. Hence, $0 \notin h^{-1}[F]$.*
  - *If $x, y \in h^{-1}[F]$ we have: $x = h^{-1}(a), y = h^{-1}(b)$ for some $a, b \in F$, then $a \wedge b \in F$ and then $a \wedge b = h(x) \wedge h(y) = h(x \wedge y)$. It follows that $x \wedge y = h^{-1}(a \wedge b) \in h^{-1}[F]$.*
  - *Let $x \in h^{-1}[F]$ and $y \in L$ such that $x \leq y$. Then $\exists b \in F$ such that $x = h^{-1}(b)$, which means that $b = h(x) \leq h(y)$. Therefore $h(y) \in F$ because $b \in F$, and then $y \in h^{-1}[F]$.*

2) If h is dense, then for any filter F in L, $h[F]$ generates a filter G in M. In fact, just take $G = \{x \in M \mid y \leq x \quad \text{for some} \quad y \in h[F]\}$. Since h is dense, $0 \notin h[F]$. If $x \in h[F]$, consider the set $B = \{y \in h[F] \mid y \leq x\} \subseteq h[F]$. We therefore have $x = \bigvee B$.

3) If h is dense, onto and F is a filter in L, then $h[F]$ is a filter in M. In fact:

  - *Since $1 \in F$, then $h(1) = 1 \in h[F]$ so that $h[F] \neq \varnothing$.*

- *Since F is a filter, $\forall a \in F, a \neq 0$ and by denseness, this implies that $\forall a \in F, h(a) \neq 0$ so that $0 \notin h[F]$.*
- *If $x, y \in h[F]$ we have : $x = h(a), y = h(b)$ for some $a, b \in F$, then $a \wedge b \in F$ and then $x \wedge y = h(a) \wedge h(b) = h(a \wedge b) \in h[F]$.*
- *Let $x \in h[F]$ and $y \in M$ such that $x \leq y$. Since h is onto, $\exists b \in L$ such that $y = h(b)$. Let $a \in F$ such that $x = h(a)$, then $a \vee b \in F$ because $a \vee b \in L, a \in F$ and $a \leq a \vee b$, then $y = x \vee y = h(a) \vee h(b) = h(a \vee b) \in h[F]$. Hence $y \in h[F]$.*

2.1.22 Proposition. *Let $h\colon L \to M$ be a frame homomorphism.*

1) If a filter $F \subseteq M$ converges (respectively clusters), then the filter $h^{-1}[F] \subseteq L$ converges (respectively clusters).

2) If h is dense, codense and onto, and a filter $F \subseteq L$ converges (respectively clusters), then the filter $h[F] \subseteq M$ converges (respectively clusters).

Proof.

1)
- Assume that F is a convergent filter in M and let us prove that $h^{-1}[F]$ converges in L. Let C be a cover of L, we want to prove that $C \cap h^{-1}[F] \neq \varnothing$. The fact that $\bigvee C = 1$ implies that $1 = h(\bigvee C) = \bigvee h[C]$ so that $h[C]$ is also a cover in M. Therefore, $F \cap h[C] \neq \varnothing$ and then $\exists x \in C$ such that $h(x) \in F$, which means that $x \in h^{-1}[F]$ so that $x \in h^{-1}[F] \cap C$. This proves that $C \cap h^{-1}[F] \neq \varnothing$.
- Assume that F is a clustered filter in M and let us prove that $h^{-1}[F]$ clusters in L. Let C be a cover of L, we want to prove that $C \cap (\text{sec}h^{-1}[F]) \neq \varnothing$. We have $\bigvee C = 1$ implies that $1 = h(\bigvee C) = \bigvee h[C]$ so that $h[C]$ is also a cover in M. Then $(\text{sec}F) \cap h[C] \neq \varnothing$. Therefore $\exists a \in C$ such that $h(a) \in \text{sec}F$. Now let us prove that $a \in C \cap (\text{sec}h^{-1}[F])$. Let $b \in h^{-1}[F]$, we want to prove that $b \wedge a \neq 0$. Let $t \in F$ such that $b = h^{-1}(t)$ so that $t = h(b)$, since $h(a) \in \text{sec}F$ and $t \in F$, we have $t \wedge h(a) \neq 0$, which means $h(a \wedge b) = h(b) \wedge h(a) \neq 0$. Hence, $a \wedge b \neq 0$ because h is a frame homomorphism.

2) Assume that h is dense, codense and onto.

- Assume that F is a convergent filter in L. Let us prove that $h[F]$ converges in M. Let C be a cover of M, we want to prove that $C \cap h[F] \neq \varnothing$. We have

$$\begin{aligned} 1 &= \bigvee \{c \in C\} \\ &= \bigvee \{h(c_1) \mid c_1 \in h^{-1}[C]\} \quad \text{because h is onto} \\ &= h\left(\bigvee \{c_1 \mid c_1 \in h^{-1}[C]\}\right) \\ 1 &= h\left(\bigvee h^{-1}[C]\right). \end{aligned}$$

Then $\bigvee h^{-1}[C] = 1$ because h is codense. So that $h^{-1}[C]$ is also a cover in L. Then $F \cap h^{-1}[C] \neq \varnothing$. Therefore $\exists x \in C$ such that $y = h^{-1}(x) \in F$, which means that $x = h(y) \in h[F]$ so that $x \in h[F] \cap C$. This proves that $C \cap h[F] \neq \varnothing$ and hence $h[F]$ converges in M.

- Assume that F is a clustered filter in L. Let us prove that $h[F]$ clusters in M. Let C be a cover of M, we want to prove that $C \cap (\text{sec}h[F]) \neq \varnothing$. We know that $h^{-1}[C]$ is also a cover of L. Then, $(\text{sec}F) \cap h^{-1}[C] \neq \varnothing$. Therefore $\exists a \in C$ such that $t = h^{-1}(a) \in \text{sec}F$ so that $a = h(t)$. Now let us prove that $a \in C \cap (\text{sec}h[F])$. Let $b \in h[F]$, we want to prove that $b \wedge a \neq 0$. Let $s \in F$ such that $b = h(s)$. Since $t \in \text{sec}F$ and $s \in F$, we have $t \wedge s \neq 0$. which means $h(t \wedge s) \neq 0$ because h is dense. It follows that $a \wedge b \neq 0$ because $0 \neq h(t \wedge s) = h(t) \wedge h(s) = a \wedge b$. Hence, $h[F]$ clusters in M.

□

Strict extensions induced by a set of filters [17].

In all the following section, X is a set of filters in the frame L. We set

$$s_X L = \{(a, \Sigma) \in L \times \mathcal{P}(X) \mid \forall F \in \Sigma, a \in F\}.$$

2.1.23 Definition. Let $s : s_X L \to L$ be the first projection map of $s_X L$ on L. Since L and $\mathcal{P}(X)$ are frames, then $L \times \mathcal{P}(X)$ is also a frame and the map s is called the **simple extension** of L **induced** by X.

2.1.24 Proposition. *$s_X L$ is a subframe of the product frame $L \times \mathcal{P}(X)$, where the meet and the join in the product of two frames are defined coordinate by coordinate. It means that*

$$(a, \Sigma_1) \wedge (b, \Sigma_2) = (a \wedge b, \Sigma_1 \cap \Sigma_2), \forall (a, \Sigma_1), (b, \Sigma_2) \in L \times \mathcal{P}(X).$$

and

$$\bigvee\{(a_\alpha, \Sigma_\alpha) \mid \alpha \in \Lambda\} = \left(\bigvee_{\alpha \in \Lambda} a_\alpha, \bigcup_{\alpha \in \Lambda} \Sigma_\alpha\right), \forall\{(a_\alpha, \Sigma_\alpha) \mid \alpha \in \Lambda\} \subseteq L \times \mathcal{P}(X).$$

Proof.

- $1 \in F, \forall F \in X$ so that $(1, X) \in s_X L \neq \varnothing, \forall F \in X$.
- $s_X L \subseteq L \times \mathcal{P}(X)$.
- Let (a_1, Σ_1) and (a_2, Σ_2) be elements of $s_X L$. We have

$$(a_1, \Sigma_1) \wedge (a_2, \Sigma_2) = (a_1 \wedge a_2, \Sigma_1 \cap \Sigma_2).$$

It remains to prove that $\forall F \in \Sigma_1 \cap \Sigma_2, a_1 \wedge a_2 \in F$. Let $F \in \Sigma_1 \cap \Sigma_2$, then $a_1 \in F$ and $a_2 \in F$. Since F is a filter, this implies that $a_1 \wedge a_2 \in F$. This shows that $(a_1, \Sigma_1) \wedge (a_2, \Sigma_2) \in s_X L$.

- Let $\{(a_\alpha, \Sigma_\alpha), \alpha \in \Lambda\}$ be a family of elements of $s_X L$. We have

$$\bigvee\{(a_\alpha, \Sigma_\alpha) \mid \alpha \in \Lambda\} = \left(\bigvee_{\alpha\in\Lambda} a_\alpha, \bigcup_{\alpha\in\Lambda} \Sigma_\alpha\right).$$

  It remains to prove that $\forall F \in \bigcup_{\alpha\in\Lambda} \Sigma_\alpha$, $\bigvee_{\alpha\in\Lambda} a_\alpha \in F$. If $F \in \bigcup_{\alpha\in\Lambda} \Sigma_\alpha$, then $F \in \Sigma_\beta$ for some $\beta \in \Lambda$, which means that $a_\beta \in F$. We have $a_\beta \leq \bigvee_{\alpha\in\Lambda} a_\alpha$ and since F is a filter, this implies that $\bigvee_{\alpha\in\Lambda} a_\alpha \in F$. Therefore $\forall F \in \bigcup_{\alpha\in\Lambda} \Sigma_\alpha$, $\bigvee_{\alpha\in\Lambda} a_\alpha \in F$. Hence $\bigvee\{(a_\alpha, \Sigma_\alpha) \mid \alpha \in \Lambda\} \in s_X L$.

□

2.1.25 Observation.

- *Since s is a projection map, then it is onto and since $s(x, \Sigma) = x, \forall (x, \Sigma) \in s_X L$, we get that $s(0_L, \varnothing) = 0_L$ so that s is dense.*

- *Since s is a frame homomorphism, it preserves all meet and then it has a left adjoint $s^*\colon L \to s_X L$ defined for any $a \in L$ by:*

$$\begin{aligned} s^*(a) &= \bigvee\{(x, \Sigma) \in s_X L \mid s(x, \Sigma) \leq a\} \\ &= \bigvee\{(x, \Sigma) \in s_X L \mid x \leq a\} \\ s^*(a) &= (a, \Sigma_a). \end{aligned}$$

  So that $s^*[L] = \{(x, \Sigma_x) \mid x \in L\}$.

- *The set $s^*[L]$ is closed under finite meets in $s_X L$. Let $x_1, x_2 \in s^*[L]$, we have $x_1 = s^*(a_1)$ and $x_2 = s^*(a_2)$, where $a_1, a_2 \in L$. Therefore,*

$$\begin{aligned} s^*(a_1) \wedge s^*(a_2) &= (a_1, \Sigma_{a_1}) \wedge (a_2, \Sigma_{a_2}) \\ &= (a_1 \wedge a_2, \Sigma_{a_1} \cap \Sigma_{a_2}) \\ s^*(a_1) \wedge s^*(a_2) &= (a_1 \wedge a_2, \Sigma_{a_1 \wedge a_2}). \end{aligned}$$

  In fact we have:

$$\Sigma_{a_1\wedge a_2} = \{F \in X \mid a_1 \wedge a_2 \in F\} \subseteq \{F \in X \mid a_1 \in F\} \cap \{F \in X \mid a_2 \in F\} = \Sigma_{a_1} \cap \Sigma_{a_2}.$$

  On the other hand we have:

$$\Sigma_{a_1} \cap \Sigma_{a_2} = \{F \in X \mid a_1 \in F\} \cap \{F \in X \mid a_2 \in F\} \subseteq \{F \in X \mid a_1 \wedge a_2 \in F\}$$

  because F is a filter. Therefore $\Sigma_{a_1\wedge a_2} = \Sigma_{a_1} \cap \Sigma_{a_2}$ and hence,

$$x_1 \wedge x_2 = s^*(a_1) \wedge s^*(a_2) = (a_1 \wedge a_2, \Sigma_{a_1\wedge a_2}) \in s^*[L].$$

  It follows that $s^*[L]$ is the generating set of some subframe of the frame $s_X L$ that we denote by $t_X L$.

Thus, an arbitrary element $t_X L$ is a join of some collection of elements of the generating set. An arbitrary subset of $s^*[L]$ is of the form $\{(a, \Sigma_a) \mid a \in A, \text{ for some } A \subseteq L\}$ so that the elements of $t_X L$ are joins of such elements. Therefore $t_X L = \bigvee\{\{(a, \Sigma_a) \mid a \in A\} \text{ for some } A \subseteq L\}$.

Let $t\colon t_X L \to L$ be the restriction of s to $t_X L$, since s is a dense onto frame homomorphism, then so is t and it is called the **strict extension** of L induced by X.

Convergence of classical filters in uniform frames [14].

2.1.26 Definition. A subset S of a frame L is **locally finite** if there is a cover C of L such that $\forall c \in C$, the set $S_c = \{s \in S \mid s \wedge c \neq 0\}$ is finite.

2.1.27 Definition. A subset S of a uniform frame $(L, \mathcal{A})$ is **uniformly locally finite** if there is a uniform cover C such that $\forall c \in C$, the set $S_c = \{s \in S \mid s \wedge c \neq 0\}$ is finite.

2.1.28 Definition. A frame L is **paracompact** if for every cover C of L there is a locally finite cover U of L such that $U \leq C$.

2.1.29 Definition. A frame L is **countably paracompact** if for every countable cover C of L there is a locally finite cover U of L such that $U \leq C$.

2.1.30 Definition. A frame $(L, \mathcal{A})$ is **uniformly countably paracompact** if for every countable cover C of L there is a uniformly locally finite cover U of L such that $U \leq C$.

2.1.31 Definition. A filter F in a uniform frame $(L, \mathcal{A})$ is **weakly Cauchy** if $(\text{sec}F) \cap C \neq \varnothing$ for any cover C of L.

2.1.32 Remark. *For a cover C of a uniform frame $(L, \mathcal{A})$, the set*

$$\tilde{C} = \{a \in L \mid a \lhd_{\mathcal{A}} t \text{ for some } t \in C\}$$

is also a cover of L. In fact, suppose $t \in C$, we set $T_t = \{s \in L \mid s \lhd_{\mathcal{A}} t\} \subseteq \tilde{C}$. Then $t = \bigvee T_t \leq \bigvee \tilde{C}, \forall t \in C$ so that $t \leq \bigvee \tilde{C}, \forall t \in C$ and then $1 = \bigvee C \leq \bigvee \tilde{C}$. This proves that $\bigvee \tilde{C} = 1$, hence $\tilde{C}$ is a cover of L.

2.1.33 Definition. A uniform frame $(L, \mathcal{A})$ is **uniformly paracompact** if for every cover C of L there is a uniformly locally finite cover U of L such that $U \leq C$.

2.1.34 Proposition. *A uniform frame $(L, \mathcal{A})$ is uniformly paracompact if and only if for every cover C of L, the cover $C^{<\omega}$ is uniform.*

Proof.

- Let L be a uniformly paracompact uniform frame, let us prove that for every cover C of L, the cover $C^{<\omega}$ is uniform. Let C be a cover of L, since L is uniformly paracompact, there exists a uniformly locally finite cover $U \subseteq L$ which refines C. Then, there is a uniform cover V such that the set $U_v = \{u \in U \mid v \wedge u \neq 0\}$ is finite for every $v \in V$ and then $\bigvee U_v \in U^{<\omega}$. Let $v \in V$, we have $v = v \wedge \bigvee U = v \wedge \bigvee U_v \leq \bigvee U_v$. In fact,

$$\begin{aligned} v \wedge \bigvee U_v &= v \wedge \bigvee \{u \in U \mid v \wedge u \neq 0\} \\ &= \bigvee \{v \wedge u \mid u \in U \text{ and } v \wedge u \neq 0\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} v \wedge \bigvee U &= v \wedge \bigvee \{u \in U\} \\ &= \bigvee \{v \wedge u \mid u \in U\} \\ &= \bigvee \{v \wedge u \mid u \in U \text{ and } v \wedge u = 0\} \vee \bigvee \{v \wedge u \mid u \in U \text{ and } v \wedge u \neq 0\} \\ &= \bigvee \{0\} \vee \bigvee \{v \wedge u \mid u \in U \text{ and } v \wedge u \neq 0\} \\ &= \bigvee \{v \wedge u \mid u \in U \text{ and } v \wedge u \neq 0\} \\ v \wedge \bigvee U &= v \wedge \bigvee U_v. \end{aligned}$$

Since U is a cover we have $\bigvee U = 1$ and then $v = v \wedge \bigvee U = v \wedge \bigvee U_v \leq \bigvee U_v$.
So that $\bigvee U_v \in U^{<\omega}$ and $v \leq \bigvee U_v, \forall v \in V$ and hence V refines $U^{<\omega}$. It remains to prove that $U^{<\omega}$ refines $C^{<\omega}$. Let $x \in U^{<\omega}$, then $x = x_1 \vee x_2 \vee \ldots \vee x_n$, where $x_1, x_2, \ldots, x_n \in U$ for some for some $n \in \mathbb{N}$. Since U refines C, $\exists y_1, y_2, \ldots, y_n$ elements of C such that $x_i \leq y_i$, for $i = 1, 2, \ldots, n$, so that $y = y_1 \vee y_2 \vee \ldots \vee y_n \in C^{<\omega}$ and $x_1 \vee x_2 \vee \ldots \vee x_n \leq y_1 \vee y_2 \vee \ldots \vee y_n$, which means that $x \leq y$. Hence $U^{<\omega}$ refines $C^{<\omega}$, so we have $V \leq U^{<\omega} \leq C^{<\omega}$. Therefore V refines $C^{<\omega}$ and since V is uniform, then so is $C^{<\omega}$.

- Let $(L, \mathcal{A})$ be a uniform frame such that for every cover S of L, the cover $S^{<\omega}$ is uniform. Let us prove that $(L, \mathcal{A})$ is uniformly paracompact. Let C be a cover of L, by [11, Proposition 8], the frame L is paracompact; so there is a locally finite cover U of L that refines C. Therefore there is a cover E of L such that the set $U_s = \{u \in U \mid s \wedge u \neq 0\}$ is finite for every $s \in E$, so that the cover $E^{<\omega}$ is uniform by hypothesis. Let $x \in E^{<\omega}$, so there is $p \in \mathbb{N}$ such that $x = x_1 \vee x_2 \vee \ldots \vee x_p$ with $x_i \in E$ for $i = 1, 2, \ldots, p$. Since the set U_s is finite for any $s \in E$, it follows that the set $U_i = \{u \in U \mid u \wedge x_i \neq 0\}$ is finite for $i = 1, 2, \ldots, p \in \mathbb{N}$ and so is the set $U_1 \cup U_2 \cup \ldots \cup U_p$ for $i = 1, 2, \ldots p$. Let us prove that $x \wedge t = 0$ for $t \in U \setminus U_1 \cup U_2 \cup \ldots \cup U_p$. In fact, if $t \in U \setminus U_1 \cup U_2 \cup \ldots \cup U_p$, we have $x \wedge t = \left(\bigvee_{i=1}^{n} x_i \right) \wedge t = \bigvee_{i=1}^{n} (x_i \wedge t) = 0$ because $t \in U \setminus U_1 \cup U_2 \cup \ldots \cup U_p$ means that $x_i \wedge t = 0$, for $i = 1, 2, \ldots, p$. This proves that $x \wedge t = 0$ for $t \in U \setminus U_1 \cup U_2 \cup \ldots \cup U_p$. Therefore, this shows that each element of the uniform cover $U^{<\omega}$ meets only finitely many elements of U, so that U is uniformly locally finite. Since $U \leq C$, it follows that L is uniformly countably paracompact.

$\square$

2.1.35 Remark. *A uniform frame $(L, \mathcal{A})$ is uniformly countably paracompact if and only if for every countable cover C of the frame L, the cover $C^{<\omega}$ is uniform.*

The proof uses the same reasoning with the proof of the first implication in Proposition (2.1.34) by assuming in the beginning that C is a countable cover of L.

2.1.36 Definition. A filter F in a frame L is **countably based** if it is generated by a countable set, which means a set of the form $S_n = \{s_n \mid n \in \mathbb{N}\}$.

2.1.37 Lemma. *If S generates a filter F in a frame L, we have*

$$\bigvee\{s^* \mid s \in S\} = \bigvee\{t^* \mid t \in F\}.$$

Proof. Let $A = \{s^* \mid s \in S\}$ and $B = \{t^* \mid t \in F\}$.

- If $a \in A$, then $a = s^*, s \in S \subseteq F$. Therefore $s \in B$ and then $s \leq \bigvee B$. Hence $\bigvee A \leq \bigvee B$.
- If $b \in B$, then $b = t^*, t \in F$. Since S generates F, $t = \bigvee\{y \mid y \in I \subseteq B \subseteq F\}$, then

  $$t^* = \left(\bigvee\{y \mid y \in I \subseteq B \subseteq F\}\right)^* = \bigwedge\{y^* \mid y \in I \subseteq B \subseteq F\}.$$

  This means that $\forall b \in I, \bigwedge\{y^* \mid y \in I \subseteq B \subseteq F\} \leq b^* \leq \bigvee\{t^* \mid t \in F\}$. Hence $\bigvee B \leq \bigvee A$.

It follows that $\bigvee B = \bigvee A$. □

2.1.38 Proposition. *If a uniform frame is uniformly countably paracompact, then every countably-based weakly Cauchy filter in it is clustered.*

Proof.

Let $(L, \mathcal{A})$ be a uniformly countably paracompact uniform frame and let F be a countably-based weakly Cauchy filter in L. Then F admits a countable base $V = \{v_n \mid n \in \mathbb{N}\}$. We want to prove that F is clustered. Assume that F is not clustered therefore by Theorem (2.1.15) and Lemma (2.1.37) we get $1_L = \bigvee\{t^* \mid t \in F\} = \bigvee\{v_n^* \mid n \in \mathbb{N}\}$. And the set $U = \{v_n^* \mid n \in \mathbb{N}\}$ is a countable cover of L. By Remark (2.1.35), $U^{<w}$ is a uniform cover of L.
Now let us prove that F cannot be weakly Cauchy, this means that there is a uniform cover W such that (sec$F$) $\cap W = \varnothing$.
Assume that (sec$F$) $\cap U^{<w} \neq \varnothing$. Then $\exists x \in$ (sec$F$) $\cap U^{<w}$ and this means the following:

- $x = (\bigvee B)^*$ such that $B \subseteq_f U$. We can therefore write $x = \bigvee\{v_{n_1}^*, v_{n_2}^*, \ldots v_{n_k}^*\}, k \in \mathbb{N}$, where $v_{n_i} \in V$ for $i = 1, 2, \ldots, k$.
- $x \in$ sec$F$ means that $\forall t \in F, x \wedge t \neq \varnothing$.

Since F is generated by V, we have $V \subseteq F$ and

$$F = \{a \in L \mid \exists p \in \mathbb{N} \text{ such that } b_{n_i} \in V \text{ for } 1 \leq i \leq p \text{ and } a \geq b_{n_1} \wedge b_{n_2} \wedge \ldots \wedge b_{n_p}\}.$$

This implies that $v_{n_1} \wedge v_{n_2} \wedge \ldots \wedge v_{n_k} \in F$ because

$$L \ni v_{n_1} \wedge v_{n_2} \wedge \ldots \wedge v_{n_k} \geq b_{n_1} \wedge b_{n_2} \wedge \ldots \wedge b_{n_k}.$$

Let $y = v_{n_1} \wedge v_{n_2} \wedge \ldots \wedge v_{n_k} \in F$ and let us prove that $x \wedge y = 0$. We have $y \leq v_{n_i}$ for $i = 1, 2, \ldots, k$, which means that $v^*_{n_i} \leq y^*$ for $i = 1, 2, \ldots, k$ and then $\bigvee\{v^*_{n_i} \mid i = 1, 2, \ldots, k\} \leq y^*$. This is equivalent to say that

$(\bigvee\{v^*_{n_i} \mid i = 1, 2, \ldots, k\}) \wedge y \leq y^* \wedge y = 0$. Hence $(\bigvee\{v^*_{n_i} \mid i = 1, 2, \ldots, k\}) \wedge y = x \wedge y = 0$.

This is a contradiction because $x \in \text{sec}F$, therefore $(\text{sec}F) \cap U^{<w} = \varnothing$. Take $W = U^{<w}$. Then F is not a weakly Cauchy filter, this is to say, F does not cluster implies that F is not weakly Cauchy and this statement is equivalent to: F is weakly Cauchy implies that F clusters.

□

2.2 Convergence of general filters

In the previous section, we have studied the notions of convergence and strong convergence in terms of filters and covers in a frame. We observe that the characteristic functions of those filters are bounded meet semi-lattices from L to 2 and then they are just a special case of general filters. In this section we formulate what those convergence notions mean in terms of general filters.

Adjoint category of the frames category.

Let **SLat** be the category of bounded meet semi-lattices and bounded meet semi-lattices homomorphisms. Let L be a frame, since every frame is a bounded meet semi-lattice, then we have the forgetful functor $E\colon$ **Frm** $\hookrightarrow$ **SLat**, which forgets the join structure of a frame (it is an embedding functor). Its left adjoint is the functor $\mathfrak{D}\colon$ **SLat** $\to$ **Frm**, which maps any bounded meet semi-lattice S to the frame $\mathfrak{D}S$ of all nonempty downsets contained in S. So $\mathfrak{D}S = \{A \subseteq S \mid \downarrow A = A \neq \varnothing\}$ and for every frame homomorphism $h\colon L \to M$, $\mathfrak{D}h\colon \mathfrak{D}L \to \mathfrak{D}M$ defined by : $\mathfrak{D}h(A) = \downarrow h(A) = \bigcup\{\downarrow (x) \mid x \in A\}, \forall A \in \mathfrak{D}L$.
We have that **SLat**$(L, ET) \cong$ **Frm**$(\mathfrak{D}L, T)$ this means that for every bounded meet semilattice homomorphism $s\colon L \to T$, there is a corresponding frame homomorphism $h\colon \mathfrak{D}L \to T$.

General filters.

2.2.1 Definition. [3] A **general filter** or a T**-valued filter** on a frame L is a bounded meet-semi lattice homomorphism $h\colon L \to T$. To make a difference, the type of filters defined above are called **classical filters**.

2.2.2 Example. *For a frame L, the map $\downarrow\colon L \to \mathfrak{J}L$, which maps an element $x \in L$ to the ideal $\downarrow x$ is a general filter on L.*

2.2.3 Definition. A general filter $\varphi\colon L \to T$ is **convergent** (or it **converges**) if for every cover C of L, $\varphi[C]$ is a cover of T.

2.2.4 Example. *For a frame L, the identity map $id_L\colon L \to L$ is a convergent filter on L.*

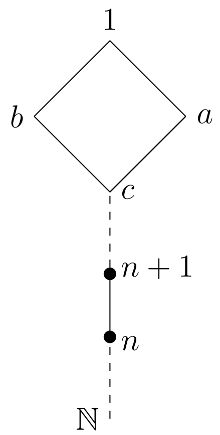
2.2.5 Definition. A general filter $\varphi\colon L \to T$ is **strongly convergent** (or it **strongly converges**) if there is a frame homomorphism $h\colon L \to T$ such that $h \leq \varphi$.

2.2.6 Definition. A **classical filter** F in a frame L can be identified with its **characteristic function** which is a general filter defined by $\mathcal{X}_F\colon L \to 2$ such that $\mathcal{X}_F(a) = 1$ if and only if $a \in F, \forall a \in L$.

2.2.7 Remark. *Consider a classical filter F in a frame L, and its characteristic function $\mathcal{X}_F\colon L \to 2$. F is convergent if every cover C of L meets F, this means that $\exists a \in F \cap C$ and then $\mathcal{X}_F(a) = 1$. Therefore $\bigvee\{\mathcal{X}_F(c) \mid c \in C\} = 1$, which actually means that $\mathcal{X}_F$ converges as a general filter on L.*

2.2.8 Remark. *[7] Every general filter which converges strongly converges but the converse does not always hold as it is shown in the example below. In fact, If $\varphi\colon L \to T$ is a general filter which strongly converges, then there is a frame homomorphism h such that $h \leq \varphi$. Now let C be a cover of L, therefore $1 = h(\bigvee C) = \bigvee h[C] \leq \bigvee \varphi[C]$. Hence, $\bigvee \varphi[C] = 1$ therefore φ converges.*

2.2.9 Example. *[7, p.458] Let us consider the frame L constructed as follows: we consider the chain $\mathbb{N} = 0 < 1 < 2 < \ldots < n < \ldots$ and then we adjoin an element c such that $\forall n \in \mathbb{N}, n < c$. Then c becomes part of the diamond with vertices $a, c, b, 1$, with 1 as the top element, a and b both bigger that c, but a and b are not comparable as it is represented below.*



Now any non trivial cover of L (different from the trivial cover $\{1\}$) must contain both a and b. Let us consider the filter $\downarrow\colon L \to \mathfrak{J}L$, which maps each $x \in L$ to $\downarrow x$.

- *The filter* $\downarrow$ *converges in* L. *In fact, for every non trivial cover* C *of* L, *we have* $\{a, b\} \subseteq C$, *which implies that* $\downarrow a \cup \downarrow b \subseteq \ \downarrow C$. *This implies that* $\{a, b\} \subseteq \ \downarrow a \cup \downarrow b \subseteq \ \downarrow C$, *therefore* $1 = \bigvee\{a, b\} \leq \bigvee(\downarrow a \cup \downarrow b) \leq \bigvee(\downarrow C)$. *This means that* $\bigvee(\downarrow C) = 1$ *for every non trivial cover* C, *hence,* $\downarrow$ *is a convergent filter.*

- *Now let us prove that* $\downarrow$ *does not strongly converge. Suppose that it does, then there is a frame homomorphism* $h\colon L \to \mathfrak{J}L$ *such that* $h \leq \downarrow$. *Therefore* $h(a) \subseteq \downarrow a$ *and* $h(b) \subseteq \downarrow b$. *Assume that* $h(a) \neq \downarrow a$, *this implies that* $h(a) \subseteq \downarrow c \subseteq \downarrow b$, *so that* $h(a)$ *and* $h(b)$ *are contained in* $\downarrow b$ *and then* $\downarrow b \supseteq h(a) \cup h(b) = h(a \vee b) = h(1_L) = 1_{\mathfrak{J}L} = \downarrow 1_L$ *and this is impossible because* $b \leq 1$, *hence* $h(a) = \downarrow a$. *Now assume that* $h(b) \neq \downarrow b$, *this implies that* $h(b) \subseteq \downarrow c \subseteq \downarrow a$, *so that* $h(a)$ *and* $h(b)$ *are contained in* $\downarrow a$ *and then* $\downarrow a \supseteq h(a) \cup h(b) = h(a \vee b) = h(1_L) = \downarrow 1_L$ *and this is impossible because* $a \leq 1$, *hence* $h(b) = \downarrow b$. *Finally, since* $a \wedge b = c$, *then we have that*

$$h(c) = h(a \wedge b) = h(a) \wedge h(b) = \downarrow a \wedge \downarrow b = \downarrow (a \wedge b) = \downarrow c.$$

On the other hand, $c = \bigvee\{x \mid x < c\}$, *so that*

$$h(c) = h(\bigvee\{x \mid x < c\}) = \bigcup\{h(x) \mid x < c\} \subseteq \bigcup\{\downarrow x \mid x < c\} \ \text{ because } \ h \leq \downarrow .$$

This implies that $\downarrow c \subseteq \bigcup\{\downarrow x \mid x < c\}$, *which means that* $c < c$. *This is impossible, hence the filter* $\downarrow$ *does not strongly converge in* L.

2.2.10 Proposition. *[7, p.458-459] For general filters on a frame* L, *the notions of convergence and strong convergence coincide for the regular and supercompact frames.*

Proof.

- If L is regular, assume that $\varphi\colon L \to T$ is a convergent filter on L. Let us consider the map $\varphi^\circ\colon L \to T$ defined by $\varphi^\circ(a) = \bigvee\{\varphi(x) \mid x \prec a\}$. Let us prove that φ° is a frame homomorphism that is below φ.
  - First of all let us prove that φ° is a general filter on L :
    * $\varphi^\circ(0) = \bigvee\{\varphi(x) \mid x \prec 0\} = \varphi(0) = 0_T$ because φ preserves the bottom.
    * $\varphi^\circ(1) = \bigvee\{\varphi(x) \mid x \prec 1\} = \varphi(1) = 1_T$ because $1 \prec 1$ and φ preserves the top.
    * If $a, b \in L$, then $\varphi^\circ(a \wedge b) = \bigvee\{\varphi(z) \mid z \prec a \wedge b\} \leq \varphi^\circ(a)$ and $\varphi^\circ(a \wedge b) \leq \varphi^\circ(b)$. This is because $z \prec a \wedge b \leq a$ and $z \prec a \wedge b \leq b$ implies that $z \prec a$ and $z \prec b$ respectively. Hence $\varphi^\circ(a \wedge b) \leq \varphi^\circ(a) \wedge \varphi^\circ(b)$. On the other hand,

      $$\begin{aligned} \varphi^\circ(a) \wedge \varphi^\circ(b) &= \bigvee\{\varphi(x) \mid x \prec a\} \wedge \bigvee\{\varphi(y) \mid y \prec b\} \\ &= \bigvee\{\varphi(x) \wedge \varphi(y) \mid x \prec a \text{ and } y \prec b\} \end{aligned}$$

      $$\varphi^\circ(a) \wedge \varphi^\circ(b) = \bigvee\{\varphi(x \wedge y) \mid x \prec a \text{ and } y \prec b\} \text{ because } \varphi \text{ preserves finite meets.}$$

      We have $\bigvee\{\varphi(x \wedge y) \mid x \prec a \text{ and } y \prec b\} \leq \bigvee\{\varphi(z) \mid z \prec a \wedge b\} = \varphi^\circ(a \wedge b)$ because $x \prec a$ and $y \prec b$ imply that $x \wedge y \prec a \wedge b$. Therefore

      $$\varphi^\circ(a) \wedge \varphi^\circ(b) \leq \varphi^\circ(a \wedge b).$$

Hence, the map φ° preserves finite meets.

– Secondly, let us prove that φ° preserves all joins. Let $S \subseteq L$ be a subset of L, and set $t = \bigvee S$. Let us consider an element $x \in L$ such that $x \prec t$, we have $\bigvee(S \cup \{x^*\}) = x^* \vee t = 1$, which means that $S \cup \{x^*\}$ is a cover of L. Now let C be a cover of L, by Remark(2.1.32), then the set $\tilde{C} = \{a \in L \mid a \prec t \text{ for some } t \in C\}$ is also a cover of L (the proof is the same for $\lhd_{\mathcal{A}} = \prec$). Since $S \cup \{x^*\}$ is a cover in L, we have

$$1 = \bigvee\{a \in L \mid a \prec s \text{ for some } s \in S \cup \{x^*\}\}.$$

Since φ is convergent, we have

$$\begin{aligned} 1 &= \bigvee\{\varphi(a) \mid L \ni a \prec s \text{ for some } s \in S \cup \{x^*\}\} \\ &= \bigvee\{\varphi^\circ(s) \mid s \in S \cup \{x^*\}\} \\ 1 &= \bigvee \varphi^\circ(S) \vee \varphi^\circ(x^*). \end{aligned}$$

We have that
$\{\varphi(y) \mid y \prec x^*\} \subseteq \{\varphi(y) \mid y \leq x^*\} \subseteq \{\varphi(y) \mid \varphi(y) \leq \varphi(x^*)\}$, which implies that $\varphi^\circ(x^*) = \bigvee\{\varphi(y) \mid y \prec x^*\} \leq \bigvee\{\varphi(y) \mid \varphi(y) \leq \varphi(x^*)\} = \varphi(x^*)$, hence $\varphi^\circ(x^*) \leq \varphi(x^*)$. Furthermore, we have $x \wedge x^* = 0$, therefore

$$\varphi(x) \wedge \varphi(x^*) = \varphi(x \wedge x^*) = \varphi(0_L) = 0_T$$

, so that $\varphi(x^*) \leq \varphi(x)^*$. Then we get $1 = \bigvee \varphi^\circ(S) \vee \varphi^\circ(x^*) \leq \bigvee \varphi^\circ(S) \vee \varphi(x)^*$, so that $\bigvee \varphi^\circ(S) \vee \varphi(x)^* = 1$, which means that $\varphi(x) \prec \bigvee \varphi^\circ(S)$ and implies that $\varphi(x) \leq \bigvee \varphi^\circ(S)$. Then, $\varphi^\circ(t) = \bigvee\{\varphi(x) \mid x \prec t\} \leq \bigvee \varphi^\circ(S)$, therefore $\varphi^\circ(\bigvee S) \leq \bigvee \varphi^\circ(S)$. On the other hand, $s \in S$ implies that $s \leq t$ and then $\varphi^\circ(s) \leq \varphi^\circ(t) = \varphi^\circ(\bigvee S)$, so that $\bigvee \varphi^\circ(S) = \bigvee\{\varphi^\circ(s) \mid s \in S\} \leq \varphi^\circ(t) = \varphi^\circ(\bigvee S)$. Finally we obtain that $\varphi^\circ(\bigvee S) = \bigvee \varphi^\circ(S)$. Hence φ° preserves all joins.

– Let $a \in L$, we have :

$$\begin{aligned} \varphi(a) \wedge \varphi^\circ(a) &= \varphi(a) \wedge \bigvee\{\varphi(x) \mid x \prec a\} \\ &= \bigvee\{\varphi(a) \wedge \varphi(x) \mid x \prec a\} \\ &= \bigvee\{\varphi(a \wedge x) \mid x \prec a\} \\ &= \bigvee\{\varphi(x) \mid x \prec a\} \\ \varphi(a) \wedge \varphi^\circ(a) &= \varphi^\circ(a). \end{aligned}$$

Hence $\varphi^\circ \leq \varphi$, so that the general filter φ is strongly convergent.

- If L is supercompact, then every cover of L contains 1 and therefore for any filter $\varphi\colon L \to T$ and for every cover C of L, $\bigvee \varphi[C] = \varphi(1) = 1$, which means that every filter in L converges. Let $e \in L$ be the largest element below 1_L, then $\{1, e\} = \uparrow e \cong 2$. Let us call λ the frame isomorphism between $\uparrow e$ and 2, and $\gamma\colon L \to \uparrow e$ is defined by $\gamma(x) =$

$x \vee e, \forall x \in L$. Therefore if $\sigma\colon 2 \to T$ be the obvious frame homomorphism, then we get a frame homomorphism $h\colon L \to \uparrow e \cong 2 \to T$, where $h = \sigma\lambda\gamma$ and is defined by $h(x) = 0_T$ if and only if $x \leq e, \forall x \in L$. If $x \in L$, there are two possibilities: if $x = 1$, then $h(1_L) = \varphi(1_L) = 1_T$. Else we have $x \leq e$ and then $h(x) = 0_T \leq \varphi(x)$. Therefore $h \leq \varphi$ on L, hence φ strongly converges.

□

2.2.11 Lemma. *Let $h\colon M \to L$ and $f\colon N \to L$ be two frame homomorphisms. If there is an onto frame homomorphism $g\colon M \to N$ such that $h = fg$, then $f_* = gh_*$. Furthermore, if h is strict, then so is f.*

Proof.

1) Let us prove that $f_* = gh_*$, which means that $\forall a \in N, b \in L, a \leq f_*(b)$ if and only if $a \leq gh_*(b)$. Let $a \in N, b \in L$.

   – If $a \leq f_*(b)$, then $f(a) \leq b$. Since g is onto, it follows that $a = gg_*(a)$, then $f(gg_*(a)) \leq b$, which implies that $hg_*(a) \leq b$ therefore, $g_*(a) \leq h_*(b)$ and finally, $a \leq gh_*(b)$.
   – If $a \leq gh_*(b)$, then $f(a) \leq fgh_*(b)$ because f is a frame homomorphism. So $f(a) \leq hh_*(b) \leq b$, then $f(a) \leq b$, which implies that $a \leq f_*(b)$.

2) Let us prove that if h is strict, then so is f. Assume that h is strict. This means that $h_*[L]$ generates M. Let us prove that $f_*[L]$ generates N. Let $a \in N$, since g is onto, $\exists x \in M$ such that $a = g(x)$ then $x = \bigvee\{b \in B \text{ for some } B \subseteq h_*[L]\} = \bigvee\{h_*(c) \mid c \in C \subseteq L\}$ therefore

$$a = g(x) = g\left(\bigvee\{h_*(c) \mid c \in C \subseteq L\}\right) = \bigvee\{gh_*(c), c \in C \subseteq L\} = \bigvee\{f_*(c) \mid c \in C \subseteq L\},$$

so that $f_*[L]$ generates N.

□

Strict extension induced by a set of general filters.

Let X be a set of general filters on a frame L and T_φ the codomain of each $\varphi \in X$. We want to construct a strict extension of L associated to X as in [6].

Let us consider the frame homomorphism $k\colon \mathfrak{D}L \to L \times \prod_{\varphi\in X} T_\varphi$ defined by

$$k(U) = \left(\bigvee U, (\overline{\varphi}(U))_{\varphi\in X}\right), \forall U \in \mathfrak{D}L,$$

and we set $\tau_X L = k[\mathfrak{D}L]$ the image of $\mathfrak{D}L$ under k, which is a subframe of $L \times \prod_{\varphi\in X} T_\varphi$. Let $\tau\colon \tau_X L \to L$ be the restriction of the projection map such that $\tau\colon \tau_X L \hookrightarrow L \times \prod_{\varphi\in X} T_\varphi \to L$.

Let g be the map $g\colon \mathfrak{O}L \to \tau_X L = k[\mathfrak{O}L]$ such that $g(U) = k(U), \forall U \in \mathfrak{O}L$. The map g thus defined is onto. We have $\forall U \in \mathfrak{O}L, \tau g(U) = \tau((\bigvee U, (\overline{\varphi}(U))_{\varphi\in X})) = \bigvee U$. This means that $\tau g = \bigvee$, which implies by Lemma (2.2.11) and Remark(1.1.30) that $\tau_* = g \bigvee_* = g\downarrow$. Since $\forall a \in L, \downarrow a \in \mathfrak{O}L$, we have

$$\begin{aligned}
\tau_*(a) &= g \downarrow a \\
&= k(\downarrow a) \\
&= \left(\bigvee \downarrow a, (\overline{\varphi}(\downarrow a))_{\varphi\in X}\right) \\
\tau_*(a) &= (a, (\varphi(a))_{\varphi\in X})\,.
\end{aligned}$$

2.2.12 Definition. The map τ thus defined is called the **strict extension** of L **induced** by X.

2.2.13 Remark. *This construction generalises the one in [17]. In fact, if X is a set of classical filters in a frame L, then every filter $F \in X$ can be identified as a general filter $\mathcal{X}_F\colon L \to 2$, having 2 as codomain and defined by $\mathcal{X}_F(a) = 1$ if and only if $a \in L$. Therefore*

$$L \times \prod_{F\in X} T_{\mathcal{X}_F} = L \times \prod_{F\in X} 2,$$

but $\prod_{F\in X} 2 = \{0,1\}^X \cong 2^X \cong \mathcal{P}(X)$. This implies that $L\times\prod_{F\in X} T_{\mathcal{X}_F} \cong L\times\mathcal{P}(X)$. Therefore $\forall a \in L, \tau_*(a) = (a, (\mathcal{X}_F(a))_{F\in X}) = (a, \Sigma_a)$ where $\Sigma_a = \{F \in X \mid a \in F\} \in \mathcal{P}(X)$ and we get $\tau_*[L] = \{(a, \Sigma_a) \mid a \in L\}$. Then, $\tau_X L$ is just the subframe of $L \times \mathcal{P}(X)$ (that we identified as $L \times \prod_{F\in X} T_{\mathcal{X}_F}$ by an isomorphism) generated by $\tau_*[L]$, which is defined by $t_X L$ in [17] and the map τ is defined there by t.

2.2.14 Lemma. *[7] Let $\varphi\colon L \to T$ be a general filter and $h\colon L \to T$ a frame homomorphism. If $G \subseteq L$ is a generating set such that $h(a) \le \varphi(a), \forall a \in G$, then $h(x) \le \varphi(x), \forall x \in L$.*

Proof. Assume that $h(a) \le \varphi(a)$ for each a in a generating set $G \subseteq L$ and set $S = \{s \in L \mid h(s) \le \varphi(s)\}$. Let us prove that $L = S$, In fact,

- We have $h(0) = 0 = \varphi(0)$ and $h(1) = 1 = \varphi(1)$. Therefore $0, 1 \in S$.
- If $x, y \in S$, then $h(x) \le \varphi(x)$ and $h(y) \le \varphi(y)$, which implies that

  $$h(x) \wedge h(y) = h(x \wedge y) \le \varphi(x) \wedge \varphi(y) = \varphi(x \wedge y).$$

  Hence, $x \wedge y \in S$.
- Let B be any subset of S, $h\left(\bigvee B\right) = \bigvee\{h(t) \mid t \in B\} \le \bigvee\{\varphi(t) \mid t \in B\} \le \varphi(\bigvee B)$ (because φ is a filter).
- $G \subseteq S$ because $h \le \varphi$ on G.

Therefore S is a subframe of L containing G. Since G is a generating set in L, L is the smallest subframe on L containing G, hence $L \subseteq S$, which means that $L = S$. □

2.2.15 Proposition. *[5] If X is a collection of general filters on a frame L, then for all $\rho \in X$, there is a unique frame homomorphism $\hat{\rho}\colon \tau_X L \to T_\rho$ such that $\hat{\rho}\tau_* = \rho$.*

Proof. Let us consider the inclusion map $j\colon \tau_X L \hookrightarrow L \times \prod_{\varphi \in X} T_\varphi$ and the projection map $pr_\rho\colon L \times \prod_{\varphi \in X} T_\varphi \to T_\rho$. We set $\hat{\rho} = pr_\rho j$, let us prove that $\hat{\rho}\tau_* = \rho$.

If $a \in L$, we have:

$$\begin{aligned}
\hat{\rho}\tau_*(a) &= \hat{\rho}\left(a, (\varphi(a))_{\varphi \in X}\right) \\
&= pr_\rho j(a, (\varphi(a))_{\varphi \in X}) \\
&= pr_\rho\left(a, (\varphi(a))_{\varphi \in X}\right) \\
\hat{\rho}\tau_* &= \rho(a).
\end{aligned}$$

This proves the existence of such a map. Now let us prove the unicity of the frame homomorphism $\hat{\rho}$. Assume there is another onto frame homomorphism $\hat{\alpha}$ such that $\hat{\alpha}\tau_* = \rho$. Then, $\hat{\alpha}\tau_* = \rho = \hat{\rho}\tau_*$. This means that $\hat{\alpha}[\tau_*[L]] = \hat{\rho}[\tau_*[L]]$ and then $\hat{\alpha}$ and $\hat{\rho}$ are equal on the set $\tau_*[L]$ which is a generating set of $\tau_X L$ (because τ is a strict extension of L) therefore by Lemma (2.2.14), $\hat{\alpha} = \hat{\rho}$ on the whole set $\tau_X L$. Hence, the map $\hat{\rho}$ is unique.

□

2.2.16 Proposition. *Let $h\colon M \to L$ be a strict extension of a frame L such that $\forall \rho \in X$, there is a frame homomorphism $\hat{\rho}\colon M \to T_\rho$ with $\hat{\rho}h_* = \rho$, therefore there is a unique onto map $\hat{h}\colon M \to \tau_X L$ such that $\tau\hat{h} = h$, which also means that $\tau_* = \hat{h}h_*$.*

Proof. Let us consider the map $v\colon M \to L \times \prod_{\varphi \in X} T_\varphi$ defined by $v(x) = (h(x), (\hat{\varphi}(x))_{\varphi \in X})$.This map is a frame homomorphism as each of its component (as a product of frame homomorphisms). Now let $x \in M$, we want to prove that $v[M] = \tau_X L$, let us first prove that $v[h_*[L]] = \tau_*[L]$:

- Let us prove that $v[h_*[L]] = \tau_X L$. In fact, for an element x of L we get:

$$\begin{aligned}
v(h_*(x)) &= (hh_*(x), (\hat{\varphi}h_*(x))_{\varphi \in X}) \\
&= (hh_*(x), (\varphi(x))_{\varphi \in X}) \\
&= (x, (\varphi(x))_{\varphi \in X}) \text{ (because h is onto)} \\
v(h_*(x)) &= \tau_*(x).
\end{aligned}$$

- Let us prove that $v[M] = \tau_X L$. In fact, let $y \in M$; since h is strict, we have

$$y = \bigvee \{h_*(c) \mid c \in C \subseteq L\}$$

  and then

$$\begin{aligned}
v(y) &= v\left(\bigvee \{h_*(c) \mid c \in C \subseteq L\}\right) \\
&= \bigvee \{vh_*(c) \mid c \in C \subseteq L\} \\
v(y) &= \bigvee \{\tau_*(c) \mid c \in C \subseteq L\} \ \in \tau_X L.
\end{aligned}$$

So that $v[M] \subseteq \tau_X L$. Now let $d \in \tau_X L$, since $\tau\colon \tau_X L \to L$ is a strict extension, then $\tau_*[L]$ generates $\tau_X L$ so that

$$\begin{aligned} d &= \bigvee\{\tau_*(a) \mid a \in A \subseteq L\} \\ &= \bigvee\{vh_*(a) \mid a \in A \subseteq L\} \\ d &= v\left(\bigvee\{h_*(a) \mid a \in A \subseteq L\}\right) \in v[M]. \end{aligned}$$

- Now, let us prove that $v\left(\bigvee\{h_*(a) \mid a \in A \subseteq L\}\right) \in v[M]$. In fact, since $h_*(a) \in M, \forall a \in A$ and M is a frame, it follows that $\bigvee\{h_*(a) \mid a \in A \subseteq L\} \in M$ and then $\tau_X L \subseteq v[M]$.

- Let us consider the map $v_1\colon M \to \tau_X L = v[M]$ such that $v_1(m) = v(m), \forall m \in M$. Then the map v_1 is an onto frame homomorphism and

$$v_1[h_*[L]] = v[h_*[L]] = \tau_*[L].$$

  So we can take $\hat{h} = v_1$.

Finally, let us prove the uniqueness of such a map. Assume that there is another onto frame homomorphism $i\colon M \to \tau_X L$ such that $\tau_* = ih_*$. Then, $\tau_*[L] = i[h_*[L]] = \hat{h}[h_*[L]]$ and since h is strict, $h_*[L]$ generates M so i and h are equal on a generating set and then by Lemma (2.2.14) we have $i[L] = \hat{h}[L]$.

□

2.2.17 Definition. Let $\mathfrak{N}$ be a set of covers of L. An $\mathfrak{N}$-**Cauchy filter** is a filter $\varphi\colon L \to T$ such that $\forall U \in \mathfrak{N}$, $\varphi(U)$ is again a cover of L.

2.2.18 Lemma. *For a set $\mathfrak{N}$ of covers of L and a collection X of general filters on L,*

$$\tau_*\colon L \to \tau_X L$$

is an $\mathfrak{N}$-Cauchy filter if and only if every filter in X is $\mathfrak{N}$-Cauchy.

Proof. For any cover C of L, we have:

$$\begin{aligned} \bigvee \tau_*[C] &= \bigvee\{\tau_*(c) \mid c \in C\} \\ &= \bigvee\{(c, (\varphi(c))_{\varphi \in X}) \mid c \in C\} \\ &= \left(\bigvee_{c \in C} c, \left(\bigvee_{c \in C} \varphi(c)\right)_{\varphi \in X}\right) \\ &= \left(1_L, \left(\bigvee \varphi[C]\right)_{\varphi \in X}\right). \end{aligned}$$

This proves that $\bigvee \tau_*[C] = 1_{\tau_X L}$ if and only if

$$\left(1_L, \left(\bigvee \varphi[C]\right)_{\varphi\in X}\right)_{\varphi\in X} = 1_{\tau_X L} = \left(1_L, \left(1_{T_\varphi}\right)_{\varphi\in X}\right), \forall\varphi \in X.$$

This condition is satisfied if and only if $\bigvee \varphi[C] = 1_{T_\varphi}, \forall\varphi \in X$, which is to say if and only if every filter $\varphi \in X$ is $\mathfrak{N}$-Cauchy. □

2.2.19 Lemma. *Let $\mathfrak{A}$ be a set of frames. Assume that X is the set of $\mathfrak{N}$-Cauchy filters $\varphi\colon L \to T_\varphi$ such that $\forall\varphi \in X, T_\varphi \in \mathfrak{A}$, then for $\mathfrak{N}^* = \{\tau_*[C] \mid C \in \mathfrak{N}\}$, every $\mathfrak{N}^*$-Cauchy filter $\rho\colon L \to T_\rho$ such that $T_\rho \in \mathfrak{A}$ is strongly convergent.*

Proof. Suppose

$$Y = \{\varphi\colon M \to T_\varphi \mid T_\varphi \in \mathfrak{A} \text{ and } \varphi \text{ is } \mathfrak{N}\text{-Cauchy filter }\}$$

and $Y^* = \{\varphi\colon M \to T_\varphi \mid T_\varphi \in \mathfrak{A} \text{ and } \varphi \text{ is } \mathfrak{N}^* \text{-Cauchy filter }\}$, where M is any frame.

Let $Y \ni \psi\colon \tau_X L \to T_\psi$, then the map $\gamma = \psi\tau_*\colon L \to T_\psi$ belongs to Y^*. Since by Lemma (2.2.19), the set $\mathfrak{N}^* = \{\tau_*[C] \mid C \in \mathfrak{N}\}$ is a set of covers of $\tau_X L$ and ψ is $\mathfrak{N}^*$-Cauchy, then $\psi\tau_*[C]$ is also a cover of T_ψ. Therefore by Proposition (2.2.16), there is a unique frame homomorphism $\hat{\gamma}\colon \tau_X L \to T_\psi$ such that $\hat{\gamma}\tau_* = \gamma = \psi\tau_*$. Let $a \in \tau_X L$, since τ is strict, then $\tau_*[L]$ generates $\tau_X L$ and then $a = \bigvee \tau_*[B] = \bigvee\{\tau_*(b) \mid b \in B \subseteq L\}$. Therefore

$$\begin{aligned}
\hat{\gamma}(a) &= \hat{\gamma}(\bigvee \tau_*[B]) \\
&= \bigvee \hat{\gamma}(\tau_*[B]) \\
\hat{\gamma}(a) &= \bigvee \psi\tau_*[B] \leq \psi\left(\bigvee \tau_*[B]\right) = \psi(a) \quad \text{(because } \psi \text{ is a filter).}
\end{aligned}$$

Then, $\hat{\gamma} \leq \psi$ and hence, ψ is strongly convergent. □

3. Applications of convergence in frames

Here, we use the convergence and clustering defined in the previous chapter to give some characterisations of frames. This chapter is mostly based on [7],[12],[14], [17] and [22], where we took our notations.

3.1 Characterising compact frames

3.1.1 Proposition. *The following properties are equivalent in a frame L :*

1) L is almost compact.

2) Every filter in L clusters.

3) Every maximal filter in L converges.

Proof.

- 1) implies 2) follows from Theorem(2.1.15) together with Lemma (2.1.17).
- 2) implies 3) follows from Property (2.1.13).
- Now let us prove that 3) implies 1). Assume that every maximal filter in L converges and L is not almost compact. Therefore there is a cover C of L such that $(\bigvee K)^* \neq \varnothing$ for any finite subset $K \subseteq C$. We set $G = \{x \in L \mid x \geq (\bigvee K)^* \text{ for some finite } K \subseteq C\}$. We have shown in the proof of Lemma (2.1.17) that G is a proper filter in L and so it is contained in an ultrafilter F in L. By hypothesis, F converges and so F meets every cover of L and then C in particular. Thus, $\exists c \in F \cap C$. Since $c \in C$, then $\{c\}$ is a finite subset of C and $c^* \geq c^* = (\bigvee\{c\})^*$ so that $c^* \in G$. It follows that $c^* \in F$ because $G \subseteq F$. Since $c \in F$, then $c \wedge c^* = 0 \in F$, which contradicts the fact that $0 \notin F$ and then L is almost compact. Hence, 3) implies 1).

□

3.1.2 Proposition. *The following properties are equivalent in a regular frame L :*

1) L is compact.

2) Every filter in L clusters.

3) Every maximal filter in L converges.

Proof. It follows from Lemma (2.1.18) and Proposition (3.1.1). □

3.1.3 Definition. A frame L is **zero dimensional** if $C(L)$ generates L. This means that $\forall x \in L, \exists B \subseteq C(L)$ such that $x = \bigvee B$.

3.1.4 Lemma. *A Boolean filter $F \subseteq L$ is maximal if and only if $(\text{sec}F) \cap C(L) \subseteq F$.*

Proof.

- Let F be a Boolean filter such that $(\text{sec}F) \cap C(L) \subseteq F$. Let G be another Boolean filter such that $F \subseteq G$. Let us prove that $F = G$. Since $F \subseteq G$, then $\text{sec}G \subseteq \text{sec}F$, so that

$$G \cap C(L) \subseteq (\text{sec}G) \cap C(L) \subseteq (\text{sec}F) \cap C(L) \subseteq F,$$

  and then $G \cap C(L) \subseteq F$. Since G is Boolean filter, it is generated by $G \cap C(L)$, this means that G is the smallest filter containing $G \cap C(L)$, then $G \cap C(L) \subseteq F$ implies that $G \subseteq F$ and then $G = F$.

- If F is a maximal Boolean filter, assume that $a \in (\text{sec}F) \cap C(L)$. Let us consider the subset $G \subseteq L$ defined by

$$G = \{y \in L \mid \exists x \in F \text{ such that } y \geq a \wedge x\}.$$

  Let us prove that G is a proper filter in L.

  - If $0 \in G$, then $0 \geq a \wedge x$ for some $x \in F$. It follows that $a \wedge x = 0$ for some $x \in F$ and this contradicts the fact that $a \in \text{sec}F$. Hence, $0 \notin G$.
  - If $c, d \in G$, then $c \geq a \wedge x$ and $d \geq a \wedge y$ for some $x, y \in F$. It follows that $c \wedge d \geq (a \wedge x) \wedge (a \wedge y) = a \wedge (x \wedge y)$ therefore $c \wedge d \geq a \wedge (x \wedge y)$. Since $x, y \in F$, then $x \wedge y \in F$ and hence $c \wedge d \in G$.
  - If $s \in G$ and $t \in L$ such that $s \leq t$, then $\exists x \in F$ such that $s \geq a \wedge x$, which implies that $t \geq a \wedge x$ because $t \geq s$ and hence, $t \in G$.

  Therefore G is a proper filter in L.
  Let $x \in F$, then $x \geq a \wedge x$ so that $x \in G$ and hence, $F \subseteq G$. Since $a \geq a = a \wedge 1$ and $1 \in F$, then $a \in G$. Now let us prove that G is a Boolean filter in L. Assume that $t \in G$, then $\exists x \in F$ such that $t \geq a \wedge x$. Since F is a Boolean filter in L and $x \in F$, then $\exists y \in F \cap C(L) \subseteq F \subseteq G$ such that $y \leq x$. It follows that $t \geq a \wedge x \geq a \wedge y$ and then $t \geq a \wedge y$. Since $a, y \in C(L)$, then $a \wedge y \in C(L)$ and since $a, y \in G$, then $a \wedge y \in G$. Hence, G is a Boolean filter in L. Since $F \subseteq G$ and F is a maximal Boolean filter in L, then $F = G$ and then $a \in F$ so that $(\text{sec}F) \cap C(L) \subseteq F$.

$\square$

3.1.5 Lemma. *Let B be a base for a frame L. A filter F converges in L if and only if for every subset $S \subseteq B$ such that $\bigvee S = 1$, $F \cap S \neq \varnothing$.*

Proof.

- Assume that F is a convergent filter on L and let $S \subseteq B$ such that $\bigvee S = 1$. Then, $S \subseteq B \subseteq L$ and $\bigvee S = 1$ imply that S is a cover of L. It follows that F meets S because F converges in L.

- Assume that for every subset $S \subseteq B$ such that $\bigvee S = 1$, $F \cap S \neq \varnothing$ and let us prove F converges in L.

  Let $C = \{x_\gamma \mid \gamma \in \Gamma\}$ be a cover of L. Since B generates L, we get

$$\begin{aligned} x_1 &= \bigvee S_1 \\ x_2 &= \bigvee S_2 \\ &\vdots \\ x_\gamma &= \bigvee S_\gamma \\ &\vdots \end{aligned}$$

  where $S_\gamma \subseteq B, \forall \gamma \in \Gamma$. Therefore, $x_\gamma = \bigvee S_\gamma, \forall \gamma \in \Gamma$ and then

$$\begin{aligned} 1 &= \bigvee C \\ &= \bigvee \{x_\gamma \mid \gamma \in \Gamma\} \\ &= x_1 \vee x_2 \vee \ldots \vee x_\gamma \vee \ldots \\ &= (\bigvee S_1) \vee (\bigvee S_2) \vee \ldots (\bigvee S_\gamma) \vee \ldots \\ 1 &= \bigvee \left(\bigcup_{\gamma \in \Gamma} S_\gamma \right). \end{aligned}$$

  Since $\bigcup_{\gamma \in \Gamma} S_\gamma \subseteq B$, then F meets $\bigcup_{\gamma \in \Gamma} S_\gamma$ by hypothesis. It follows that $\exists x \in F \cap \bigcup_{\gamma \in \Gamma} S_\gamma$, which means that $x \in \bigcup_{\gamma \in \Gamma} S_\gamma$. It means that $x \in S_{\gamma_0}$ for some $\gamma_0 \in \Gamma$ so that

$$x \leq \bigvee S_{\gamma_0} = x_{\gamma_0} \in C$$

  and since $x \in F$, it follows that $x_{\gamma_0} \in F$. Then, $x_{\gamma_0} \in F \cap C$ and hence, F converges in L.

$\square$

3.1.6 Proposition. *The following properties are equivalent in a zero-dimensional frame L:*

1) L is a compact frame.

2) Every Boolean filter in L clusters.

3) Every maximal Boolean filter in L converges.

Proof.

- Let L be a compact frame. Then, by Proposition (3.1.2) every filter in L cluters. So in particular every Boolean filter in L clusters. Hence 1) implies 2).

- Assume that every Boolean filter in L clusters and let M be a maximal Boolean filter. Then, (sec$M$) $\cap C(L) \subseteq M$ by Lemma (3.1.4). To prove that M is convergent, it suffices by Lemma (3.1.5) to prove that for every subset $S \subseteq C(L)$ such that $\bigvee S = 1, M \cap S \neq \varnothing$. Let us consider a subset $S \subseteq C(L)$ such that $\bigvee S = 1$. Then, (sec$M$)$\cap C(L) \cap S \subseteq M \cap S$ because (sec$M$)$\cap C(L) \subseteq M$. Since $S \subseteq C(L)$, then (sec$M$)$\cap C(L) \cap S =$ (sec$M$)$\cap S$ and it follows that (sec$M$)$\cap S \subseteq M \cap S$. Since M clusters by hypothesis, then (sec$M$)$\cap S \neq \varnothing$ so that $\varnothing \neq$ (sec$M$) $\cap S \subseteq M \cap S$ and hence, $M \cap S \neq \varnothing$. This proves that M converges and hence 2) implies 3).

- Assume that every maximal Boolean filter in L converges and L is not compact. So there is a cover C of L which does not have a finite subcover. Now let

$$S = \{s \in C(L) \mid s \leq c \text{ for some } c \in C\}.$$

  L is a zero-dimensional frame, which means that every element of L is a join of complemented elements below it. Then for any $c \in C$, there is a subset $S_c \subseteq S$ such that $c = \bigvee S_c$. We therefore have that, for each $s \in S, c = \bigvee S_c \leq \bigvee S$. Consequently, c is an upper bound for the set S, which then implies that $1 = \bigvee C \leq \bigvee S$, hence $\bigvee S = 1$, showing that S is a cover of L. The cover S has no finite subcover. In fact, if S had a finite subcover, say $D = \{d_1, \ldots, d_n\}$, then for each i we could find $c_i \in C$ such that $d_i \leq c_i$ and so we would have $1 = d_1 \vee \ldots \vee d_n \leq c_1 \vee \ldots \vee c_n$, therefore $c_1 \vee \ldots \vee c_n = 1$, which would imply that C has a finite subcover $T = \{c_1, \ldots, c_n\}$ and this would contradict the assumption that L is not compact. Therefore L is compact and hence 3) implies 1).

□

3.2 Characterising spatial frames

3.2.1 Proposition. *[12] A regular frame L is spatial if and only if*

$$\forall x \in L \ \textit{ such that } x \neq 1, \exists p \in \Sigma L \ \textit{ such that } x \leq p.$$

Proof. Let L be a regular frame.

- If the frame L is spatial, assume that $a \in L \setminus \{1\}$, then there exists $I \subseteq \Sigma L$ such that $I \neq \varnothing$ and $a = \bigwedge\{p \mid p \in I\}$, this means that $a \leq p$, for every $p \in I$ and we are done.

- Now assume that $\forall x \in L$ with $x \neq 1, \exists p \in \Sigma L$ with $x \leq p$ and let us prove that L is spatial, this means that $\forall a, b \in L$ such that $a \nleq b, \exists p \in \Sigma L$ with $a \leq p$, but $b \nleq p$. Let

$x, y \in L$ such that $x \not\leq y$. Since L is regular, $x = \bigvee\{a \in L \mid a \prec x\}$ and $x \leq y$ means that $\bigvee\{a \in L \mid a \prec x\} \leq y$. This also means that $\forall a \in L$ such that $a \prec x, a \leq y$. Therefore the negation of this statement ($x \not\leq y$) is equivalent to saying that $\exists b \in L$ such that $b \prec x$ and $b \not\leq y$. The fact that $b \not\leq y$ implies that $b \not\prec y$, which is equivalent to $b^* \vee y \neq 1$. By hypothesis, $\exists p \in \Sigma L$ such that $b^* \vee y \leq p$, which implies that $y \leq b^* \vee y \leq p$ and hence $y \leq p$. On the other hand, $x \not\leq p$. In fact, if $x \leq p$, then $1 = b^* \vee x \leq p$ because $b \prec x$ and this contradicts the assumption that $p \neq 1$. Hence the frame L is spatial.

□

3.2.2 Proposition. *Let L be a frame, then L is spatial if and only if every filter which clusters in L strongly clusters in L.*

Proof.

- If L is spatial, let F be a filter in L which clusters. Then, $\bigvee\{x^* \in L \mid x \in F\} \neq 1$ by Theorem (2.1.15). By Proposition (3.2.1), spatiality implies that $\bigvee\{x^* \in L \mid x \in F\} \leq p$ for some $p \in \Sigma L$. Hence F strongly clusters.
- If every filter which clusters in L strongly clusters in L, assume that $a \in L \setminus \{1\}$. We set $S = \{s^* \mid s \prec a\} \subseteq L$, which is a base for a filter F in L. In fact:
  - If $0 \in S$, then $0 = x^*$, for some $x \prec a$, which means that $1 = x^* \vee a = 0 \vee a = a$, this contradicts the fact that $a \neq 1$. Hence $0 \notin S$.
  - If $b_1, b_2 \in S$, then $b_1 = s_1^*$ and $b_2 = s_2^*$ for some $s_1, s_2 \in L$ such that $s_1 \prec a$ and $s_2 \prec a$. Therefore

    $b_1 \wedge b_2 = s_1^* \wedge s_2^* = (s_1 \vee s_2)^* \in S$ because $s_1 \prec a$ and $s_2 \prec a$ imply that $s_1 \vee s_2 \prec a$.

  Now let us prove that F is clustered. By Theorem (2.1.15), it suffices to prove that $\bigvee\{x^* \mid x \in F\} \neq 1$.
  First of all, let us prove that $a = \bigvee\{x^{**} \in L \mid x \prec a\}$. Since L is regular,

  $$a = \bigvee\{x \in L \mid x \prec a\}$$

  and it suffices to prove that $\bigvee\{x \in L \mid x \prec a\} = \bigvee\{x^{**} \in L \mid x \prec a\}$:

  - if $t \in L$ such that $t \prec a$, then $t \leq t^{**} \leq \bigvee\{x^{**} \in L \mid x \prec a\}$, this implies that

    $$t \leq \bigvee\{x^{**} \in L \mid x \prec a\}, \text{ for every } t \in L \text{ such that } t \prec a.$$

    Hence $\bigvee\{x \in L \mid x \prec a\} \leq \bigvee\{x^{**} \in L \mid x \prec a\}$.
  - If $y = b^{**}$, where $b \in L$ is such that $b \prec a$, then $y = b^{**} \prec a$. This implies that $y \in \{x \in L \mid x \prec a\}$ and then $y \leq \bigvee\{x \in L \mid x \prec a\}$. Therefore

    $$\bigvee\{x^{**} \in L \mid x \prec a\} \leq \bigvee\{x \in L \mid x \prec a\}.$$

Hence $a = \bigvee\{x^{**} \in L \mid x \prec a\} = \bigvee\{x \in L \mid x \prec a\}$.
Secondly, let us note that the proper filter F is such that

$$F = \{y \in L \mid y \geq x_1 \wedge \ldots \wedge x_n \text{ for some finitely many } x_i \in S\}.$$

Then if $t \in F$, $t \geq t_1^* \wedge t_2^* \wedge \ldots \wedge t_p^* = (t_1 \vee t_2 \vee \ldots \vee t_p)^*$ for some $p \in \mathbb{N}$ and each of $t_1, t_2, \ldots, t_p$ is an element of L which is rather below a. Since $t_1 \vee t_2 \ldots \vee t_p \prec a$, then $(t_1 \vee t_2 \vee \ldots \vee t_p)^* \in S$. Let $y = t_1 \vee t_2 \ldots \vee t_p \prec a$. Then, $t \geq y^*$, which implies that $t^* \leq y^{**}$. Consequently, $t^* \leq y^{**} \leq \bigvee\{x^{**} \in L \mid x \prec a\}, \forall t \in F$ so that $\bigvee\{t^* \in L \mid t \in F\} \leq \bigvee\{x^{**} \in L \mid x \prec a\}$. Also, if $s \in S \subseteq F$, then $s = u^*$ for some $u \prec a$ and then $u^{**} = s^* \leq \bigvee\{t^* \in L \mid t \in F\}$ so that $\bigvee\{s^* \in L \mid s \in S\} \leq \bigvee\{t^* \in L \mid t \in F\}$. Hence by Lemma (2.1.37) we get:

$$\bigvee\{u^{**} \in L \mid u \prec a\} = \bigvee\{s^* \in L \mid s \in S\} \leq \bigvee\{t^* \in L \mid t \in F\}.$$

Finally we get that

$$a = \bigvee\{u^{**} \in L \mid u \prec a\} \leq \bigvee\{t^* \in L \mid t \in F\} \leq \bigvee\{x^{**} \in L \mid x \prec a\} = a.$$

Hence $\bigvee\{t^* \in L \mid t \in F\} = a \neq 1$, which means that the filter F clusters in L and by hypothesis it strongly clusters so that there exists $p \in \Sigma L$ such that

$$a = \bigvee\{t^* \in L \mid t \in F\} \leq p,$$

therefore by Proposition (3.2.1) L is spatial.

□

3.3 Characterising uniformly paracompact uniform frames

The following theorem characterises uniformly paracompact uniform frames in terms of clustering of weakly Cauchy filters.

3.3.1 Theorem. *[14, Proposition 3.1] A uniform frame $(L, \mathcal{A})$ is uniformly paracompact if and only if every weakly Cauchy filter in L clusters.*

Proof.

- Let $(L, \mathcal{A})$ be a uniformly paracompact uniform frame. Let us prove that every weakly Cauchy filter in L clusters. Let F be a weakly Cauchy filter in L, then sec$F$ meets every cover in L. Let C be a cover in L then $C^{<\omega}$ is a cover of L. Let us prove that sec$F \cap C \neq \varnothing$. Since $C^{<\omega}$ is a cover of L, we have sec$F \cap C^{<\omega} \neq \varnothing$. Let $x \in C^{<\omega}$, we have $x = \bigvee B$ for some $B \subseteq_f C$, then $\forall y \in F$, we have $x \wedge y \neq 0$, which means that $\bigvee B \wedge y = \bigvee_{b \in B}(b \wedge y) \neq 0$. This implies that $\exists t \in B$ such that $\forall y \in F, t \wedge y \neq 0$ because B is finite. It follows that $t \in$ sec$F$. Consequently, $t \in C \cap$ sec$F$ so that $C \cap$ sec$F \neq \varnothing$. Hence F clusters.

- Now let $(L, \mathcal{A})$ be uniform frame in which every weakly Cauchy filter clusters and let S be a cover of L. Let us prove that $S^{<\omega}$ is a uniform cover. Let us consider the set $\widetilde{S} = \{x \in L \mid x \prec a \text{ for some } a \in S\}$. There are two possibilities:
  - Assume that for some $C \subseteq_f S, \bigvee C = 1$, then $1 \in S^{<\omega} = \{\bigvee A \mid A \subseteq_f S\}$ and $S^{<\omega}$ is a uniform cover in L.
  - Assume that $\bigvee C \neq 1$ for every $C \subseteq_f S$, then set $M = \{(\bigvee A)^* | A \subseteq_f \widetilde{S}\}$. Then M generates the proper filter defined by:

    $$F = \{y \in L \mid y \geq x_1 \wedge \ldots \wedge x_n \text{ for some finitely many } x_i \in M\}.$$

    In fact, if m_1 and m_2 are elements of M, then $m_1 = (\bigvee B_1)^*$ and $m_2 = (\bigvee B_2)^*$, where B_1 and B_2 are two finite subsets of $\widetilde{S}$. We have $\bigvee(B_1 \cup B_2) = (\bigvee B_1) \vee (\bigvee B_2)$ so $(\bigvee(B_1 \cup B_2))^* = (\bigvee B_1)^* \wedge (\bigvee B_2)^* = m_1 \wedge m_2$. Since B_1 and B_2 are finite, then $B_1 \cup B_2$ is also a finite subset of $\widetilde{S}$ and hence $m_1 \wedge m_2 \in M$. This proves that M is closed under finite meets. If $0 \in M$, then $(\bigvee B)^* = 0$ for some $B \subseteq_f \widetilde{S}$, which implies that $(\bigvee B) = (\bigvee B)^{**} = 1$. This contradicts the hypothesis and hence $0 \notin M$.

    Now let us prove that F is not clustered in L. Let s be an element of $\widetilde{S}$, then $\{s\} \subseteq_f \widetilde{S}$ so that $(\bigvee\{s\})^* = s^* \in M \subseteq F$. This means that $\forall s \in \widetilde{S}, s^* \in F$. But we cannot have $s \in F$ because this will imply that $s^* \wedge s = 0 \in F$ and this contradicts the assumption that $0 \notin F$. Therefore, F is not clustered and then cannot be weakly Cauchy by hypothesis. This means that there is a uniform cover $D \subseteq L$ such that $D \cap \text{sec}F = \varnothing$. Let $d \in D \setminus \text{sec}F$. This means that $d \wedge y = 0$ for some $y \in F$. Since $y \in F$, there exists $x_1, x_2, \ldots, x_n \in M$ such that $y \geq x_1 \wedge x_2 \wedge \ldots \wedge x_n$ so that each $x_i = (\bigwedge B_i)^*, B_i \subseteq_f \widetilde{S}$. Let $p_i = \bigwedge B_i$. Then, $0 = d \wedge y \geq d \wedge (p_1^* \wedge p_2^* \wedge \ldots \wedge p_n^*)$, which means that $d \wedge (p_1^* \wedge p_2^* \wedge \ldots \wedge p_n^*) = 0$ and it follows that $d \wedge (p_1 \vee p_2 \vee \ldots \vee p_n)^* = 0$ because $p_1^* \wedge p_2^* \wedge \ldots \wedge p_n^* = (p_1 \vee p_2 \vee \ldots \vee p_n)^*$. Since every $p_i \in \widetilde{S}$ because every B_i is finite, then there exists $z_i \in \widetilde{S}$ such that $p_i \prec z_i$ for $i = 1, 2, \ldots m$. Therefore $p_1 \vee p_2 \vee \ldots \vee p_m \prec z_1 \vee z_2 \vee \ldots \vee z_m$, which implies that

    $$(p_1 \vee p_2 \vee \ldots \vee p_m)^{**} \prec z_1 \vee z_2 \vee \ldots \vee z_m.$$

    Finally, we get $d \leq (p_1 \vee p_2 \vee \ldots \vee p_n)^{**} \leq z_1 \vee z_2 \vee \ldots \vee z_m \in S^{<\omega}$. Then D refines $S^{<\omega}$, hence $S^{<\omega}$ is a uniform cover of L which is then a uniformly paracompact frame.

□

3.4 $\mathbb{F}$-compactness

In the previous sections, we have given characterisations of compactness in terms of all classical ultrafilters converging and then we need to understand what compactness will look like if we consider a more general type of filters. Since we do know what convergence means for general filters, we use this concept of convergence to formulate the kind of compactness that fits with general filters. In this section, unlike the first one, the compactness notion here is defined in terms of imposing a specific type of general filters to converge. It turns out that this new notion of compactness characterises compact frames. This section is mostly based on paper [7], where we took most of the notations.

3.4.1 Definition. A **lax retract** of a frame M is a frame L such that we can find some frame homomorphisms $\alpha\colon L \to M$ and $\beta\colon M \to L$ such that $\beta\alpha \leq id_L$.

3.4.2 Definition. A function $\mathbb{F}$ between objects of the category **Frm** is a **filter selection** if for every frame L, the set $\mathbb{F}(L) = \{$bounded - meet semilattice homomorphisms $\varphi\colon L \to X$ with $X \in$ **Frm**$\}$ is a collection of filters satisfying the following properties:

- Every frame homomorphism $h\colon L \to M$ belongs to $\mathbb{F}(L)$.
- For any two filters $\varphi\colon L \to M$ in $\mathbb{F}(L)$ and $\rho\colon M \to L$ in $\mathbb{F}(M)$, $\rho\varphi \in \mathbb{F}(L)$.

We can also define a *filter selection* as follows: Let S be the category of all frames and bounded-meet semilattice homomorphisms. Any filter selection $\mathbb{F}$ yields a subcategory $S(\mathbb{F})$, where objects are those of **Frm** and morphisms (arrows) are exactly the filters $\varphi\colon L \to X$ belonging to the filter selection $\mathbb{F}(L)$ with $\mathbb{F}(L) = \{$bounded - meet semilattice homomorphisms $\varphi\colon L \to X$ with $X \in$ **Frm**$\}$. By definition, we have **Frm** $\subseteq S(\mathbb{F}) \subseteq S$.

On the other hand, any subcategory $D \subseteq S$ in which the objects are those of **Frm** and morphisms (arrows) the elements $\varphi\colon L \to X$ that are exactly elements of the filter selection $\mathbb{F}(L)$ yields a filter selection $\mathbb{F}_D$ defined for every $L \in$ **Frm** by $\mathbb{F}_D(L) = \bigcup\{D(L,X) \mid X \in$ **Frm**$\}$. Where $D(L,X)$ is a filter defined from L to X. This shows a one-one correspondence between $\mathbb{F} \mapsto S(\mathbb{F})$ and $D \mapsto \mathbb{F}_D$. Therefore we can identify filter selections as subcategories of S such that **Frm** $\subseteq S$.

3.4.3 Definition. Let $\mathbb{F}$ be a filter selection. A frame L is $\mathbb{F}$- **compact** (respectively **strongly** $\mathbb{F}$- **compact)** if every filter in $\mathbb{F}(L)$ converges (respectively strongly converges).

3.4.4 Remark. *Every strongly $\mathbb{F}$-compact frame is $\mathbb{F}$-compact because every strongly convergent filter converges. But the converse does not always hold. In fact, suppose that*

$$\mathbb{F}(L) = \{\varphi\colon L \to X \mid X \in obj(S) \ \textit{ and } \ \varphi \ \textit{ converges}\},$$

so that L is $\mathbb{F}$-compact. But if there is a filter $\varphi \in \mathbb{F}(L)$, which does not converge, then L cannot be strongly $\mathbb{F}$-compact. The notions of $\mathbb{F}$-compactness and strongly $\mathbb{F}$-compactness can only coincide in a frame L, where every general filter which is convergent strongly converges, for example in regular frames.

3.4.5 Definition. An $\mathbb{F}$-**lax retract of a frame** M is a frame L such that we can find some frame homomorphism $h\colon L \to M$ and a filter $\varphi\colon M \to L$ in $\mathbb{F}(M)$ such that $\varphi h \leq id_L$.

Every lax-retract is an $\mathbb{F}$-lax retract because every frame homomorphism is a filter, but the converse does not hold as shown in the example below.

3.4.6 Example. *Let L be a non-trivial frame, then there is a unique frame homomorphism $h\colon 2 \to L$ defined by $h(0) = 0$ and $h(1) = 1$. Let $\mathbb{F}$ be a filter selection such that $\mathbb{F}(L)$ contains the classical filter $\rho\colon L \to 2$ defined by $\rho(a) = 1$ if and only if $a \in \{1\} = F$. Therefore we have $\rho h(a) = a, \forall a \in 2$ and this proves that 2 is an $\mathbb{F}$-lax retract of L. But in case the frame L has no frame homomorphism $h\colon L \to 2$, 2 cannot be a lax retract of L.*

3.4.7 Proposition. *For a filter selection $\mathbb{F}$, if L is an $\mathbb{F}$-compact (respectively strongly $\mathbb{F}$-compact) frame, then we have the following :*

1) *For every $a \in L$, the frame $\uparrow a$ is $\mathbb{F}$-compact (respectively strongly $\mathbb{F}$-compact).*
2) *Every $\mathbb{F}$-lax retract of L is $\mathbb{F}$-compact (respectively strongly $\mathbb{F}$-compact).*

Proof.

- Assume that L is an $\mathbb{F}$-compact frame:

  1) Suppose that $a \in L$ and let $\rho\colon \uparrow a \to X$ be an element of $\mathbb{F}(\uparrow a)$. Then the filter $\rho\lambda\colon L \to X$ belongs to $\mathbb{F}(L)$, where the frame homomorphism $\lambda\colon L \to \uparrow a$ is defined by $\lambda(x) = a \vee x, \forall x \in L$ and then $\rho\lambda$ converges by $\mathbb{F}$-compactness of L. Then for every cover C of $\uparrow a$ we have:
$$\begin{aligned} 1_{\uparrow a} &= 1_L \\ &= \rho\lambda[C] \\ &= \rho[\{\lambda(c) \mid c \in C\}] \\ &= \rho[\{c \vee a \mid c \in C\}] \\ &= \rho[\{c \mid c \in C\}] \quad \text{because } c \in \uparrow a \\ 1_{\uparrow a} &= \rho[C]. \end{aligned}$$
     This proves that ρ is also convergent hence $\uparrow a$ is $\mathbb{F}$-compact.
  2) Let M be an $\mathbb{F}$-lax retract of L. Then there is a frame homomorphism $\alpha\colon L \to M$ and a filter $\beta\colon M \to L$ in $\mathbb{F}(M)$ such that $\beta\alpha \leq id_L$. Let $\lambda\colon L \to X$ be a filter in $\mathbb{F}(L)$, then the filter $\lambda\beta\colon M \to X$ belongs to $\mathbb{F}(M)$ and then the filter $\lambda\beta\alpha\colon L \to X$ belongs to $\mathbb{F}(L)$ and is convergent by $\mathbb{F}$-compactness of L. Furthermore, $\beta\alpha \leq id_L$ implies that $\lambda\beta\alpha \leq \lambda id_L = \lambda$ and then λ converges. In fact for a cover $C \subseteq M$ we have :
$$1_M = \bigvee\{\lambda\beta\alpha(x) \mid x \in C\} \leq \bigvee\{\lambda(x) \mid x \in C\}.$$
     Hence M is $\mathbb{F}$-compact.

- Assume that L is a strongly $\mathbb{F}$-compact frame :

  1) Suppose $a \in L$ and let $\rho\colon \uparrow a \to X$ be an element of $\mathbb{F}(\uparrow a)$. Then for the frame homomorphism $\lambda\colon L \to \uparrow a$ defined by $\lambda(x) = a \vee x, \forall x \in L$, we have
$$\mathbb{F}(L) \ni \rho\lambda\colon L \to X,$$

which is strongly convergent by strong $\mathbb{F}$-compactness of L. Then, there is a frame homomorphism $h\colon L \to X$ such that $h \leq \rho\lambda$. Therefore we have

$$h(a) \leq \rho\lambda(a) = 0.$$

In fact, $\lambda(a) = a = 0_{\uparrow a}$, which implies that $\rho\lambda(a) = \rho(0_{\uparrow a}) = 0_X$. Hence $h(a) = 0$. Therefore,

$$\begin{aligned}
\text{ker}\lambda &= \{x \in L \mid \lambda(x) = 0_{\uparrow a} = a\} \\
&\subseteq \{x \in L \mid \rho\lambda(x) = \rho(0_{\uparrow a}) = 0_X\} \text{ (because } \rho \text{ preserves } 0) \\
\text{ker}\lambda &\subseteq \{x \in L \mid h(x) = 0_X\} = \text{ker}h \text{ (because } h \leq \rho\lambda).
\end{aligned}$$

Thus $\text{ker}\lambda \subseteq \text{ker}h$, therefore there is a unique frame homomorphism $k\colon \uparrow a \to X$ such that $k\lambda = h$.

We therefore have that $\forall b \in \uparrow a$,

$$h(b) = k\lambda(b) = k(a \vee b) = k(b)$$
$$\rho\lambda(b) = \rho(a \vee b) = \rho(b),$$

then $h \leq \rho\lambda$ also holds on $\uparrow a$ and this implies that

$$\forall b \in \uparrow a, k(b) = h(b) \leq \rho\lambda(b) = \rho(b).$$

This means that $k \leq \rho$ and then ρ strongly converges, hence $\uparrow a$ is strongly $\mathbb{F}$-compact.

2) Let M be an $\mathbb{F}$-lax retract of L. Therefore there is a frame homomorphism $\alpha\colon M \to L$ and a filter $\mathbb{F}(L) \ni \beta\colon L \to M$ such that $\beta\alpha \leq id_M$. Let $\mathbb{F}(M) \ni \lambda\colon M \to X$, let us prove that λ strongly converges. We get $\mathbb{F}(L) \ni \lambda\beta\colon L \to X$, which strongly converges by strong $\mathbb{F}$-compactness of L, so there is a frame homomorphism

$$f\colon L \to X$$

such that $f \leq \lambda\beta$. This implies that $f\alpha \leq \lambda\beta\alpha$ and $\beta\alpha \leq id_M$ implies that $\lambda\beta\alpha \leq \lambda id_M = \lambda$. We therefore get $f\alpha \leq \lambda\beta\alpha \leq \lambda$, since f and α are frame homomomorphisms, then so is $f\alpha\colon M \to X$ and this proves that the filter λ strongly converges, hence M is strongly $\mathbb{F}$-compact.

□

For a frame L, and a filter selection $\mathbb{F}$, $\mathbb{F}(L)$ yields a nucleus on $\mathfrak{O}L$ defined by

$$n_{\mathbb{F}L}\colon \mathfrak{O}L \to \mathfrak{O}L,$$

$$n_{\mathbb{F}L}(U) = \bigcap\{\overline{\varphi}_*\overline{\varphi}(U) \mid \varphi \in \mathbb{F}(L)\}, \forall U \in \mathfrak{O}L,$$

where $\overline{\varphi}\colon \mathfrak{D}L \to X$ is given by Proposition (1.1.20) (we consider L as a meet semi-lattice). Note that the frame homomorphism $\overline{\varphi}\colon \mathfrak{D}L \to X$ determined by the filter $\varphi \in \mathbb{F}(L)$ is such that $\overline{\varphi}(\downarrow a) = \varphi(a), \forall a \in L$ and $\overline{\varphi}(U) = \bigvee\{\varphi(u) \mid u \in U\}, \forall U \in \mathfrak{D}X$ and $\overline{\varphi}_*\overline{\varphi}$ is the nucleus associated to the frame homomorphism $\overline{\varphi}$.

3.4.8 Lemma. *For $U \in \mathfrak{D}L$, we have $\bigvee_L n_{\mathbb{F}L}(U) = \bigvee_L U$.*

Proof.

- Since $\forall \varphi \in \mathbb{F}(L), \overline{\varphi}_*\overline{\varphi}$ is a nuclei, it implies that
$$U \subseteq \overline{\varphi}_*\overline{\varphi}(U), \forall \varphi \in \mathbb{F}(L).$$
Therefore
$$U \subseteq \bigcap\{\overline{\varphi}_*\overline{\varphi}(U) \mid \varphi \in \mathbb{F}(L)\} = n_{\mathbb{F}L}(U).$$
Hence
$$\bigvee_L U \leq \bigvee_L n_{\mathbb{F}L}(U).$$
- First, let us note that we have
$$\overline{id_L}(U) = \bigvee U, \forall U \in \mathfrak{D}L,$$
and
$$\overline{id_L}_*(a) = \downarrow a, \forall a \in L.$$
Now let $v \in n_{\mathbb{F}L}(U) = \bigcap\{\overline{\varphi}_*\overline{\varphi}(U) \mid \varphi \in \mathbb{F}(L)\}$. Since $id_L \in \mathbb{F}(L)$, we have
$$v \in \overline{id_L}_*\overline{id_L}(U) = \downarrow (\bigvee U),$$
which implies that $v \leq \bigvee U$. Hence
$$\bigvee_L n_{\mathbb{F}L}(U) \leq \bigvee_L U.$$

This proves that $\bigvee_L n_{\mathbb{F}L}(U) = \bigvee_L U$. □

We now set $\mathfrak{F}L = \{A \in \mathfrak{D}L \mid n_{\mathbb{F}L}(A) = A\}$.

Let us prove that $\mathfrak{F}L$ is a subframe of $\mathfrak{D}L$:

- $n_{\mathbb{F}L}(0_{\mathfrak{D}L}) = 0_{\mathfrak{D}L}$. In fact,
$$\begin{aligned} n_{\mathbb{F}L}(0_{\mathfrak{D}L}) &= n_{\mathbb{F}L}(\varnothing) \\ &= \bigcap\{\overline{\varphi}_*\overline{\varphi}(\varnothing) \mid \varphi \in \mathbb{F}(L)\} \text{ (this is because } \overline{\varphi} \text{ is a frame homomorpphism)} \\ &= \{\overline{\varphi}_*(0_X) \mid \varphi \in \mathbb{F}(L)\} \text{ (this is because } \overline{\varphi}_*\colon X \to \mathfrak{D}L \text{ is bottom preserving)} \\ &= \{0_{\mathfrak{D}L} \mid \varphi \in \mathbb{F}(L)\} \\ n_{\mathbb{F}L}(0_{\mathfrak{D}L}) &= 0_{\mathfrak{D}L}. \end{aligned}$$
Then $0_{\mathfrak{D}L} \in \mathfrak{F}L$.

- $n_{\mathbb{F}L}(1_{\mathfrak{O}L}) = 1_{\mathfrak{O}L}. Infact,$

$$\begin{aligned} n_{\mathbb{F}L}(1_{\mathfrak{O}L}) &= \bigcap\{\overline{\varphi}_*\overline{\varphi}(1_{\mathfrak{O}L}) \mid \varphi \in \mathbb{F}(L)\} \text{ (this is because } \overline{\varphi} \text{ is a frame homomorpphism)} \\ &= \{\overline{\varphi}_*(1_X) \mid \varphi \in \mathbb{F}(L)\} \text{ (this is because } \overline{\varphi}_* \colon X \to \mathfrak{O}L \text{ is top preserving)} \\ &= \{1_{\mathfrak{O}L} \mid \varphi \in \mathbb{F}(L)\} \\ n_{\mathbb{F}L}(1_{\mathfrak{O}L}) &= 1_{\mathfrak{O}L}. \end{aligned}$$

  Then $1_{\mathfrak{O}L} \in \mathfrak{F}L$.

- Now let $U, V \in \mathfrak{F}L$. Then,

$$\begin{aligned} n_{\mathbb{F}L}(U \cap V) &= \bigcap\{\overline{\varphi}_*\overline{\varphi}(U \cap V) \mid \varphi \in \mathbb{F}(L)\} \\ &= \bigcap\{\overline{\varphi}_*\left(\overline{\varphi}(U) \cap \overline{\varphi}(V)\right) \mid \varphi \in \mathbb{F}(L)\} \\ &= \bigcap\{\overline{\varphi}_*\left(\overline{\varphi}(U)\right) \cap \overline{\varphi}_*\left(\overline{\varphi}(V)\right) \mid \varphi \in \mathbb{F}(L)\} \\ &= \left(\bigcap\{\overline{\varphi}_*\left(\overline{\varphi}(U)\right) \mid \varphi \in \mathbb{F}(L)\}\right) \cap \left(\bigcap\{\overline{\varphi}_*\left(\overline{\varphi}(V)\right) \mid \varphi \in \mathbb{F}(L)\}\right) \\ &= n_{\mathbb{F}L}(U) \cap n_{\mathbb{F}L}(V) \\ n_{\mathbb{F}L}(U \cap V) &= U \cap V. \end{aligned}$$

  Therefore $U \cap V \in \mathfrak{F}L$.

- Let us define the join in $\mathfrak{F}L$ by :

$$\bigvee'\{A_\alpha \mid \alpha \in \Gamma\} = n_{\mathbb{F}L}\left(\bigvee\{A_\alpha \mid \alpha \in \Gamma\}\right), \forall (A_\alpha)_{\alpha\in\Gamma} \in \mathfrak{F}L,$$

  and let us prove that $\mathfrak{F}L$ is closed under the join thus defined. Let $(V_\alpha)_{\alpha\in\Gamma} \in \mathfrak{F}L$, then

$$\begin{aligned} n_{\mathbb{F}L}\left(\bigvee'\{V_\alpha \mid \alpha \in \Gamma\}\right) &= n_{\mathbb{F}L}\left(n_{\mathbb{F}L}\left(\bigvee\{V_\alpha \mid \alpha \in \Gamma\}\right)\right) \\ &= n_{\mathbb{F}L}\left(\bigvee\{V_\alpha \mid \alpha \in \Gamma\}\right) \text{ (this is because } n_{\mathbb{F}L} \text{ is a nucleus)} \\ n_{\mathbb{F}L}\left(\bigvee'\{V_\alpha \mid \alpha \in \Gamma\}\right) &= \bigvee'\{V_\alpha \mid \alpha \in \Gamma\}. \end{aligned}$$

  Then $\bigvee'\{V_\alpha \mid \alpha \in \Gamma\} \in \mathfrak{F}L$.

We have thus proved that $\mathfrak{F}L$ is a frame with the following:

- $0_{\mathfrak{F}L} = 0_{\mathfrak{O}}L$.
- $1_{\mathfrak{F}L} = 1_{\mathfrak{O}L} = L$.
- $\bigwedge_{\mathfrak{F}L} = \bigwedge_{\mathfrak{O}L} = \bigcap$.

- $\bigvee_{\mathfrak{F}L} = \bigvee' = n_{\mathbb{F}L}(\bigvee)$.

This gives us the frame homomorphism $k\colon \mathfrak{F}L \to L$ such that

$$k(U) = \bigvee_L n_{\mathbb{F}L}(U) = \bigvee_L U, \forall U \in \mathfrak{F}L.$$

In other words, if we consider the map

$$m\colon \mathfrak{D}L \to L,$$

defined by $m(U) = \bigvee_L U, \forall U \in \mathfrak{D}L$, the map k is the restriction of m map $\mathfrak{F}L$, so that we can write $k = m\diagup_{\mathfrak{F}L}$.

Now let us prove that $n_{\mathbb{F}L}(\downarrow a) = \downarrow a$.

$$\text{We have } \; n_{\mathbb{F}L}(\downarrow a) = \bigcap\{\overline{\varphi}_*\overline{\varphi}(\downarrow a) \mid \varphi \in \mathbb{F}(L)\}.$$

- Since $\overline{\varphi}_*\overline{\varphi}$ is a nucleus, we have $\downarrow a \subseteq \overline{\varphi}_*\overline{\varphi}(\downarrow a)$.
- Since $id_L \in \mathbb{F}(L)$, we have

$$\bigcap\{\overline{\varphi}_*\overline{\varphi}(\downarrow a) \mid \varphi \in \mathbb{F}(L)\} = n_{\mathbb{F}L}(\downarrow a) \subseteq \overline{id_L}_*\overline{id_L}(\downarrow a) = \downarrow (\bigvee(\downarrow a)) = \downarrow a,$$

which implies that $n_{\mathbb{F}L}(\downarrow a) \subseteq \downarrow a$. Hence $n_{\mathbb{F}L}(\downarrow a) = \downarrow a$ and this means that

$$\forall a \in L, \downarrow a \in \mathfrak{F}L.$$

3.4.9 Remark. *$\{\downarrow a \mid a \in L\}$ generates $\mathfrak{F}L$. In fact, if $U \in \mathfrak{F}L$, then $n_{\mathbb{F}L}(U) = U$, which means that*

$$n_{\mathbb{F}L}(\downarrow U) = \downarrow U = U = \bigvee\{\downarrow x \mid x \in U \subseteq L\}.$$

3.4.10 Definition. [7] A filter selection $\mathbb{F}$ is **natural** if the filter $\downarrow\colon L \to \mathfrak{F}L$ is in $\mathbb{F}(L)$.

Since $\varphi = \overline{\varphi} \downarrow, \forall \varphi \in \mathbb{F}(L)$, the second property for filter selections implies that the filter $\varphi \in \mathbb{F}(L)$ if $\downarrow\colon L \to \mathfrak{F}L$ belongs to $\mathbb{F}(L)$.

3.4.11 Proposition. *[7] The following are equivalent for any natural filter selection $\mathbb{F}$:*

1) *The frame L is strongly $\mathbb{F}$-compact.*
2) *$\downarrow\colon L \to \mathfrak{F}L$ strongly converges.*
3) *The frame L is a lax retract of $\mathfrak{F}L$.*

Proof.

- Assume that L is a strongly $\mathbb{F}$-compact frame. This means that any filter in $\mathbb{F}(L)$ is strongly convergent and then $\mathbb{F}(L) \ni \downarrow: L \to \mathfrak{F}L$ strongly converges because $\mathbb{F}$ is natural. Then 1) implies 2).
- Assume that $\downarrow: L \to \mathfrak{F}L$ strongly converges, then there is a frame homomorphism
$$h: L \to \mathfrak{F}L$$
such that $h \leq \downarrow$. This implies that $kh \leq k \downarrow = id_L$. In fact
$$k(\downarrow U) = \downarrow k(U) = \downarrow (\bigvee_L U) = U, \forall U \in \mathfrak{F}L.$$
Then 2) implies 3).
- Assume that L is a lax retract of $\mathfrak{F}L$. By Proposition (1.1.20), it is sufficient to prove that $\mathfrak{F}L$ is strongly $\mathbb{F}$-compact because this will later imply that L is strongly $\mathbb{F}$-compact.
Let us consider the filter $\varphi: \mathfrak{F}L \to X$, which belongs to $\mathbb{F}(\mathfrak{F}L)$. Then the filter
$$\rho = \varphi \downarrow: L \to X$$
belongs to $\mathbb{F}(\mathfrak{F}L)$ because $\mathbb{F}$ is natural ($\downarrow \in \mathbb{F}(L)$). Then ρ induces a frame homomorphism $\overline{\rho}: \mathfrak{D}L \to X$. Therefore we can consider the frame homomorphism $p = \overline{\rho}\diagup_{\mathfrak{F}L}: \mathfrak{F}L \to X$, which is the restriction of $\overline{\rho}$ to $\mathfrak{F}L \subseteq \mathfrak{D}L$ and defined by
$$p(U) = \overline{\rho}(U), \forall U \in \mathfrak{F}L.$$
For every $a \in L, \downarrow a \in \mathfrak{F}L$ and we get
$$p(\downarrow a) = \overline{\rho}(\downarrow a) = \rho(a) = \varphi(\downarrow a).$$

Since
$$p(\downarrow a) = \overline{\rho}(\downarrow a) = \rho(a) = \varphi(\downarrow a), \forall a \in L,$$
if we set $A = \{\downarrow x \mid x \in L\}$, we get
$$p(A) = \{p(\alpha) \mid \alpha \in A\} = \{\varphi(\alpha) \mid \alpha \in A\} = \varphi(A).$$
Therefore, p and φ coincide on a generating set and by Proposition(2.2.14) they coincide on the whole set $\mathfrak{F}L$. It implies that $\overline{\rho}\diagup_{\mathfrak{F}L} = p \leq \varphi$ on $\mathfrak{F}L$ and hence $\mathfrak{F}L$ is strongly $\mathbb{F}$-compact. □

Some particular cases of filter selections.

In general, to construct a filter selection $\mathbb{F}$, for a frame L, we need the following:

- A collection $\mathcal{A}$ of subsets of L.

- A collection of filters $\varphi\colon L \to T$ in $\mathbb{F}(L)$ satisfying the property,

$$\varphi(\bigvee S) = \bigvee \varphi[S], \forall S \in \mathcal{A}.$$

- Therefore $\mathbb{F}(L) = \{\varphi\colon L \to T \mid \varphi(\bigvee S) = \bigvee \varphi[S], \forall S \in \mathcal{A}\}$.

Now to see how this construction works, let us consider two particular cases.

- The filter selection $\mathbb{A}$ is obtained as follows:
  - Consider the collection $\mathcal{A} = \{\varnothing\}$
  - In this case, $\mathbb{A}(L) = \{\mathcal{X}_F\colon L \to 2 \mid F$ is a filter in $L\}$. In fact, we have that $\mathcal{X}_F(\bigvee \varnothing) = \bigvee \mathcal{X}_F(\varnothing) = 0$.
- The filter selection $\mathbb{P}$ is obtained as follows:
  - Consider the collection $\mathcal{A} = \{X \subseteq L \mid X$ is finite$\}$
  - In this case, $\mathbb{P}(L) = \{\mathcal{X}_F\colon L \to 2 \mid F$ is a prime filter in $L\}$. In fact, $\forall S \in \mathcal{A}$, we have that

$$\mathcal{X}_F(\bigvee S) = \bigvee \mathcal{X}_F(S) = \begin{cases} 1 \text{ if } S \cap F \neq \varnothing \\ 0 \text{ else} \end{cases}$$

    because F is prime and S is finite.

The following two propositions give characterisation of compactness in terms of $\mathbb{F}$-compactness.

3.4.12 Proposition. *A frame L is $\mathbb{A}-$compact if and only if it is supercompact.*

Proof.

- If L is $\mathbb{A}-$compact, then every filter in $\mathbb{F}(L)$ is convergent and then by [7, Proposition 5], the filter $\downarrow\colon L \to \mathfrak{F}L = \mathfrak{D}L$ belongs to $\mathbb{A}(L)$ and hence it is convergent. This means that for a cover C of L, $\downarrow C = \{\downarrow c \mid c \in C\}$ is also a cover of $\mathfrak{D}L$, which means that

$$\bigcup\{\downarrow c \mid c \in C\} = 1_{\mathfrak{D}L} = L = \downarrow 1.$$

  Therefore $1 \in \downarrow 1 = \bigcup\{\downarrow c \mid c \in C\}$ so that $\exists c \in C$ such that $1 \in \downarrow c$. Therefore $c = 1$ and then $1 \in C$, hence L is supercompact.

- If L is supercompact, then every cover of C of L contains 1, then for every bounded lattice homomorphism $\mathbb{F}(L) \ni \varphi\colon L \to T$, $1 = \varphi(1) \in \varphi[C]$. Therefore every filter in $\mathbb{F}(L)$ is convergent and hence L is $\mathbb{A}$-compact.

$\square$

3.4.13 Proposition. *A frame L is $\mathbb{P}-$compact if and only if it is compact.*

The proof of this theorem can be found in [7, Proposition 6].

4. Convergence on locales

Here we talk about convergence of filters on a locale. We are going to prove that every classical filter in a locale L induces a filter on the locale L and if the locale L is T_1, then the strong convergence of the classical filter in L is equivalent to the convergence of its induced filter on the locale L. It turns out that if we identify a filter in a locale with its induced filter on a locale (it means that we consider a filter in a frame as a filter on that same frame), then the concept of convergence on a locale generalises the concept of strong convergence in a frame. We will also show that the convergence and clustering on locales characterise sharp points and compact locales. Note that in this chapter, given a locale L, we will denote the maps J_L and r_L as j and r respectively. This chapter is based mainly on paper [13], where we took our notations.

4.1 A localic view of convergence

In this section we study the convergence of filters on locales and their properties.

4.1.1 Definition. A **filter on a locale** L is a filter in the sublattice of $\mathcal{S}\ell(L)$ of all sublocales of L.

4.1.2 Definition. A **Cartan filter** on a locale L is a filter on the Boolean algebra Sub(L) of all complemented sublocales of L.

4.1.3 Definition. An **open filter** on a locale L is a filter on the collection $\mathcal{O}(L)$ of all open sublocales of L.

4.1.4 Remark.

- *We assume that* $O \notin \mathcal{F}$ *for any filter* $\mathcal{F}$ *on a locale* L.
- *For a strict extension* $h\colon M \to L$ *of a locale* L, $h_*[L]$ *is a sublocale and we define the finite join in* $h_*[L]$ *by* $h_*(x) \sqcup h_*(y) = h_*(x \vee y), \forall x, y \in L$.

4.1.5 Definition. A **neighbourhood** of a point $p \in \Sigma L$ is any open sublocale $\mathfrak{o}(x)$ such that $x \in L$ and $p \in \mathfrak{o}(x)$.

4.1.6 Definition. A filter $\mathcal{F}$ on a locale L **converges** to a point $p \in \Sigma L$ if $p \in \mathfrak{o}(x)$ implies that $\exists S \in \mathcal{F}$ such that $S \subseteq \mathfrak{o}(x), \forall x \in L$. For an extension $h\colon M \to L$, $h^*[L]$ is a sublocale of M and then a Cartan filter $\mathcal{F}$ on $h^*[L]$ **converges** to $p \in \Sigma M$ if $p \in \mathfrak{o}(y)$ implies that $\mathfrak{o}_{h^*[L]}(h_*h(y)) \in \mathcal{F}, \forall y \in M$.

Let L be a frame and F a classical filter in L. Then F induces the following:

- A filter on L defined by $\mathcal{F}_s = \{S \in \mathcal{S}\ell(L) \mid \mathfrak{o}(s) \subseteq S \text{ for some } s \in F\}$.
- An open filter on L defined by $\mathcal{F}_o = \{\mathfrak{o}(s) \mid s \in F\}$.

- A Cartan filter on L by Sub$(L), \mathcal{F}_c = \{S \in \mathcal{S}\ell(L) \mid \mathfrak{o}(s) \subseteq S$ for some $s \in F\}$.

The following proposition shows that if L is a T_1 locale, then the convergence on L is a generalisation of the notion of strong convergence in L.

4.1.7 Proposition. *Let L be a T_1 locale and F a filter in L, then any of the above filters induced by $\mathcal{F}$ on L converges if and only if F strongly converges.*

We only prove that this proposition is true for the Cartan filter because the proofs for the other cases use the same way of thinking.

Proof.

- Assume that the filter $\mathcal{F}_c$ converges to some point $p \in \Sigma L$. Let Q be the completely prime filter defined by $Q = \{x \in L \mid x \not\leq p\}$. Let us prove that $Q \subseteq F$. Let y be an element of Q, then $y \not\leq p$ and then $p \notin \mathfrak{c}(y)$, which means that $p \in \mathfrak{o}(y)$. Since the filter $\mathcal{F}_c$ converges to p, then $\exists b \in F$ with $\mathfrak{o}(b) \in \mathcal{F}_c$ and $\mathfrak{o}(b) \subseteq \mathfrak{o}(y)$, which means that $b \leq y$ and implies that $y \in F$ because b belongs to the filter F. Therefore $Q \subseteq F$ and since Q is completely prime, then F strongly converges.

- Now suppose that F strongly converges, then F contains a completely prime filter say F_p. Let p be the point of L given by $p = \bigvee\{x \in L \mid x \notin F_p\}$. Let us prove that the filter $\mathcal{F}_c$ converges to p. Let $a \in L$ such that $p \in \mathfrak{o}(a)$, which means that $p \notin \mathfrak{c}(a) =\uparrow a$, (because $\mathfrak{o}(a) \cap \mathfrak{c}(a) = \mathsf{O}$) it also means that $a \not\leq p$, so that $a \vee p = 1$. In fact, if $a \vee p \neq 1$, then $p \leq a \vee p$ implies by maximality that $p = a \vee p$ and this contradicts the fact that $a \not\leq p$. Therefore $a \vee p = 1 \in F_p$ and since $p \notin F_p$, then $a \in F_p$ because F_p is a prime filter. It follows that $a \in F_p \subseteq F$, and then $\mathfrak{o}(a) \in \mathcal{F}$. Hence $\mathcal{F}_c$ converges to p.

□

4.1.8 Proposition. *Let $\mathcal{F}$ be any filter on a Hausdorff locale L. If $\mathcal{F}$ converges to $p \in \Sigma L$, then $\bigcap\{\overline{F} \mid F \in \mathcal{F}\} = \{p, 1\}$.*

Proof.

- Let $a \in \bigcap\{\overline{F} \mid F \in \mathcal{F}\}$. Then $a \in \overline{F} =\uparrow (\bigwedge F), \forall F \in \mathcal{F}$. We want to prove that $a \in \{p, 1\}$. Let us consider the set $A = \{t \in L \mid t \sqsubseteq p\}$, and let b be an element of A. Then $b \sqsubseteq p$, which means that $b \leq p$ and $b^* \not\leq p$ so that $p \in \mathfrak{o}(b^*)$. Since $\mathcal{F}$ converges to p, $\exists S \in \mathcal{F}$ such that $S \subseteq \mathfrak{o}(b^*)$, which means that $\uparrow (\bigwedge S) = \overline{S} \subseteq \overline{\mathfrak{o}(b^*)} = \mathfrak{c}(b^{**}) =\uparrow b^{**}$ so that $\uparrow (\bigwedge S) \subseteq\uparrow b^{**}$. Since $\bigwedge S \in\uparrow (\bigwedge S)$, it is in $\uparrow b^{**}$ and then $b \leq b^{**} \leq \bigwedge S \leq a$, because $S \in \mathcal{F}$ and $a \in\uparrow (\bigwedge S)$. Therefore $b \leq a, \forall a \in A$ and then $p = \bigvee A \leq a$ because L is a Hausdorff locale. It follows that $p \leq a$ and then $p = a$ or $p < a$. Then, $p < a$ means that $a = 1$. This is because L is Hausdorff and then it is T_1 and since p is prime, it is maximal. Therefore $a \in \{p, 1\}$, hence $\bigcap\{\overline{F} \mid F \in \mathcal{F}\} \subseteq \{p, 1\}$.

- Now let us prove that $\{p, 1\} \subseteq \bigcap\{\overline{F} \mid F \in \mathcal{F}\}$. Since 1 is in every sublocale of L, it remains to prove that $p \in \bigcap\{\overline{F} \mid F \in \mathcal{F}\}$. Suppose that $p \notin \bigcap\{\overline{F} \mid F \in \mathcal{F}\}$, then $\exists S \in \mathcal{F}$ such that $p \notin \overline{S} = \mathfrak{c}(\bigwedge S)$, it means that $p \in \mathfrak{o}(\bigwedge S)$. Since $\mathcal{F}$ converges, $\exists T \in \mathcal{F}$ such that $T \subseteq \mathfrak{o}(\bigwedge S)$, this means that $T \cap \overline{S} = T \cap \mathfrak{c}(\bigwedge S) = \mathsf{O}$. Since $T \cap S \subseteq T \cap \overline{S} = \mathsf{O}$, then $T \cap S = \mathsf{O}$, which contradicts the fact that $S, T \in \mathcal{F}$. It follows that $p \in \bigcap\{\overline{F} \mid F \in \mathcal{F}\}$ and hence $\{p, 1\} \subseteq \bigcap\{\overline{F} \mid F \in \mathcal{F}\}$.

□

4.1.9 Corollary. *There is uniqueness of limits in Hausdorff locales.*

In fact, If a filter $\mathcal{F}$ on a locale L converges to $p, q \in \Sigma L$, then by Proposition (4.1.8),

$$\bigcap\{\overline{F} \mid F \in \mathcal{F}\} = \{p, 1\} = \{q, 1\}$$

so that $p = q$.

4.1.10 Definition. For a locale L, a sublocale $S \subseteq L$ and a Cartan filter $\mathcal{F}$ on S, we say that the filter $\mathcal{F}$ **clusters** at a point $p \in \Sigma L$ if $\forall x \in L$, the fact that $p \in \mathfrak{o}(x)$ implies that $\forall F \in \mathcal{F}, \mathfrak{o}(x) \cap \mathcal{F} \neq \mathsf{O}$. This is also equivalent to saying that $p \in \bigcap\{\overline{F} \mid F \in \mathcal{F}\}$.

4.1.11 Definition. Let us consider $h\colon M \to L$ an extension of a locale L and $p \in \Sigma M$. An **ideal** $I \subseteq L$ or $I \subseteq \mathrm{coz}L$ **coconverges** to p if $\forall b \in M$ with $b \vee p = 1, \exists v \in I$ such that $h(b) \vee v = 1$.

4.1.12 Remark. *Let us consider an extension $h\colon M \to L$ of a locale L, then $h^*[L]$ is a sublocale in M. Therefore any filter $\mathcal{F}$ in $\mathcal{S}\ell(h^*[L])$ induces an ideal $I_{\mathcal{F}}$ and any ideal $I \subseteq L$ induces a filter $\mathcal{F}_I$ in $\mathcal{S}\ell(h^*[L])$ such that*

$$\mathcal{F}_I = \{T \in \mathcal{S}\ell(h^*[L]) \mid T \subseteq \mathfrak{c}(h_*(s)) \text{ for some } s \in I\}$$

and

$$I_{\mathcal{F}} = \{s \in L \mid \mathfrak{c}(h_*(s)) \in \mathcal{F}\},$$

so that $I_{\mathcal{F}_I} = I$ and $\mathcal{F}_{I_{\mathcal{F}}} \subseteq \mathcal{F}$. The following two propositions shows us in which case the convergence of the filter and the coconvergence of the associated ideal are related.

4.1.13 Proposition. *Let $h\colon M \to L$ be an extension of a locale L such that the locale M is T_1, I an ideal in L and $p \in \Sigma M$. Then I coconverges to p if and only if $\mathcal{F}_I$ converges to p.*

Proof.

- Assume that I coconverges to $p \in \Sigma M$ and let $a \in M$ such that $p \in \mathfrak{o}(a)$. Then $p \notin \mathfrak{c}(a) = \uparrow a$ so that $a \nleq p$. Since M is T_1 and $p \in \Sigma M$, therefore p is maximal and we have $p \leq a \vee p$, which implies that $a \vee p = p$ or $a \vee p = 1$, but if $a \vee p = p$, then

$a \leq p$, which contradicts the fact that $a \nleq p$, therefore $a \vee p = 1$. Since I coconverges to p, $\exists b \in I$ such that $h(a) \vee b = 1_L$, then

$$h_*h(a) \sqcup h_*(b) = h_*(h(a) \vee b) = h_*(1_L) \ = 1_{h_*[L]}.$$

Consequently, $\mathfrak{c}_{h_*[L]}(h_*(b)) \subseteq \mathfrak{o}_{h_*[L]}(h_*h(a))$. In fact, if $\exists t \in \mathfrak{c}_{h_*[L]}(h_*(b))$ and

$$t \notin \mathfrak{o}_{h_*[L]}(h_*h(a)),$$

then $t \in \mathfrak{c}_{h_*[L]}(h_*h(a)) \cap \mathfrak{c}_{h_*[L]}(h_*(b)) = \mathfrak{c}(h_*h(a)) \cap \mathfrak{c}(h_*(b)) \cap h_*[L]$ so that $t \geq h_*h(a)$ and $t \geq h_*(b)$. It follows that $t \geq h_*h(a) \sqcup h(b) = 1_{h_*[L]}$ and then $t = 1_{h_*[L]} \in \mathfrak{o}_{h_*[L]}(h_*h(a))$. This contradicts the fact that $t \notin \mathfrak{o}_{h_*[L]}(h_*h(a))$, hence $\mathfrak{c}_{h_*[L]}(h_*(b)) \subseteq \mathfrak{o}_{h_*[L]}(h_*h(a))$. Since $\mathfrak{c}_{h_*[L]}(h_*(b)) \in \mathcal{F}_I$ (because $b \in I$), therefore $\mathfrak{o}_{h_*[L]}(h_*h(a)) \in \mathcal{F}_I$, hence $\mathcal{F}_I$ converges to p.

- Assume that $\mathcal{F}_I$ converges to p and let $b \in M$ such that $b \vee p = 1$. Therefore, $b \nleq p$ because if $b \leq p$, therefore $b \vee p = p = 1$, which contradicts the fact that $1 \neq p \in \Sigma M$. Then, $p \notin \mathfrak{c}(b)$ and it follows that $p \in \mathfrak{o}(b)$. Since $\mathcal{F}_I$ converges to p, $\mathfrak{o}_{h_*[L]}(h_*h(b)) \in \mathcal{F}_I$. Therefore $\exists a \in I$ such that $\mathfrak{c}_{h_*[L]}(h_*(a)) \subseteq \mathfrak{o}_{h_*[L]}(h_*h(b))$, this implies that

$$\mathsf{O} = \mathfrak{c}_{h_*[L]}(1_{h_*[L]}) = \mathfrak{c}_{h_*[L]}(h_*(a)) \cap \mathfrak{c}_{h_*[L]}(h_*h(b)) = \mathfrak{c}_{h_*[L]}(h_*h(b) \sqcup h_*(a)).$$

Consequently, $h_*h(b) \sqcup h_*(a) = 1_{h_*[L]}$ so that $h_*(h(b) \vee a) = h_*(1_L)$ and this implies that $hh_*(h(b) \vee a) = hh_*(1_L)$. It follows that $h(b) \vee a = 1_L$ (because h is onto) and hence I coconverges to p.

□

4.1.14 Proposition. *Let $h : M \to L$ be a regular extension of a locale L, $\mathcal{F}$ a filter on $h_*[L]$ and $p \in \Sigma M$. Then, $I_{\mathcal{F}}$ coconverges to p if and only if $\mathcal{F}$ converges to p.*

Proof.

- Assume that $I_{\mathcal{F}}$ coconverges to p. Prove that $\forall m \in M$ with $m \vee p = 1$, $\exists u \in I_{\mathcal{F}}$, $h(m) \vee u = 1$.

  Since L is regular, then it is T_1 and by Proposition (4.1.13), the filter $\mathcal{F}_{I_{\mathcal{F}}}$ converges to p. Then $\forall a \in L$, $p \in \mathfrak{o}(a)$ implies that $\exists F \in \mathcal{F}_{I_{\mathcal{F}}} \subseteq \mathcal{F}$ such that $F \subseteq \mathfrak{o}(a)$, hence $\mathcal{F}$ converges to p.

- On the other hand, assume that $\mathcal{F}$ converges to p. Let $b \in M$ such that $b \vee p = 1$, then $b \nleq p$, which implies that $p \in \mathfrak{o}(b)$. By regularity, $b = \bigvee\{x \in M \mid x \prec b\} \nleq p$ and then $\exists y \in M$ such that $y \prec b$ and $y \nleq p$, therefore $p \in \mathfrak{o}(y)$ and then $\mathfrak{o}_{h_*[L]}(h_*h(y)) \in \mathcal{F}$ (because $\mathcal{F}$ converges to p) and it implies that $\overline{\mathfrak{o}_{h_*[L]}(h_*h(y))}^{h_*[L]} \in \mathcal{F}$ because

$$\mathfrak{o}_{h_*[L]}(h_*h(y)) \subseteq \overline{\mathfrak{o}_{h_*[L]}(h_*h(y))}^{h_*[L]}$$

  and $\mathfrak{o}_{h_*[L]}(h_*h(y)) \in \mathcal{F}$.

Before going ahead with the proof, let us first prove the following claim: h is dense implies that $(h_*h(y))^* = h_*h(y^*)$. In fact, assume that $b \in L$.

- We have $b^* \leq h_*h(b^*)$ and $b \leq h_*h(b)$, which imply that $(h_*h(b))^* \leq b^* \leq h_*h(b^*)$. On the other hand we have:

$$\begin{aligned} h_*h(b) \wedge h_*h(b^*) &= h_*(h(b) \wedge h(b^*)) \\ &= h_*(h(b \wedge b^*)) \\ &= h_*h(0) \\ &= h_*(0) \\ h_*h(b) \wedge h_*h(b^*) &= 0 \quad \text{(because h is dense).} \end{aligned}$$

  Hence $h_*h(b^*) \leq (h_*h(b))^*$ so that $h_*h(b^*) = (h_*h(b))^*$.

  Now coming back to our main proof gives us:

$$\begin{aligned} \overline{\mathfrak{o}_{h_*}(h_*h(y))}^{h_*[L]} &= h_*[L] \cap \overline{\mathfrak{o}(h_*h(y))} \\ &= h_*[L] \cap \mathfrak{c}((h_*h(y))^*) \\ &= h_*[L] \cap \mathfrak{c}(h_*h(y)) \in \mathcal{F} \\ &= \mathfrak{c}_{h_*[L]}(h_*h(y)) \in \mathcal{F}. \end{aligned}$$

  It then follows that $h(y) \in I_{\mathcal{F}}$. Since $y \prec b$ we get $y^* \vee b = 1$ then

$$h(y^*) \vee h(b) = h(y^* \vee b) = h(1) = 1,$$

  then $I_{\mathcal{F}}$ coconverges to p.

□

4.1.15 Proposition. *Let $h : M \to L$ be a compact Hausdorff extension of L. Then a proper ideal $I \subseteq L$ coconverges to $p \in \Sigma M$ if and only if $\bigvee\{h_*(u) \mid u \in I\} = p$.*

Proof.

- Assume that I coconverges to p. We set $s = \bigvee\{h_*(u) \mid u \in I\}$.

  First of all let us prove that $s \neq 1$. If $s = 1$, then $\{h_*(u) \mid u \in I\}$ is a cover of M and by compactness there are $v_1, v_2, \cdots, v_k \in I$ such that $h_*(v_1) \vee h_*(v_2) \vee \cdots \vee h_*(v_k) = 1$, for some $k \in \mathbb{N}$. Consequently,

$$1 = h(1) = h(h_*(v_1) \vee h_*(v_2) \vee \cdots \vee h_*(v_k)) = hh_*(v_1) \vee hh_*(v_2) \vee \cdots \vee hh_*(v_k)$$

  so that $v_1 \vee v_2 \vee \ldots \vee v_k = 1 \in I$ this is because h is onto $(hh_* = id_L)$ and I is an ideal. It follows that $I = L$, which contradicts the fact that I is a proper ideal.

  Since M is Hausdorff, we have $p = \bigvee\{a \in M \mid a \sqsubseteq p\}$. Let us prove that

$$\bigvee\{a \in M \mid a \sqsubseteq p\} = s.$$

Since M is Hausdorff, it is T_1 and then p is a maximal element in M, so it is sufficient to prove that $\bigvee\{a \in M \mid a \sqsubseteq p\} \leq s$ (and then conclude by maximality). Assume that $a \in L$ and p is a maximal element in M such that $a \nleq p$ then, $a \vee p = 1$. In fact, if $a \vee p \neq 1$ then, $p \leq a \vee p$ implies that $a \vee p = p$ or $a \vee p = 1$, we cannot have $a \vee p = p$ (which contradicts the assumption $a \nleq p$), then $a \vee p = 1$.

Since I coconverges to p, $\exists w \in I$ such that $h(a^*) \vee w = 1$, this implies that $h(a) \leq w$. In fact,

$$\begin{aligned} h(a) &= h(a) \wedge 1 \\ &= h(a) \wedge (h(a^*) \vee w) \\ &= (h(a) \wedge h(a^*)) \vee (h(a) \wedge w) \\ &= h(a \wedge a^*) \vee (h(a) \wedge w) \\ &= 0 \vee (h(a) \wedge w) \\ h(a) &= h(a) \wedge w. \end{aligned}$$

Therefore $h(a) \leq w$ and then, $a \leq h_*(w) \leq s$. Consequently, $a \leq s$, for every $a \in L$ such that $a \sqsubseteq p$. It follows that $p = \bigvee\{a \in L \mid a \sqsubseteq p\} \leq s$ and then $p \leq s$. Since $s \neq 1$ and p is maximal, we get $p = s$.

- On the other hand, suppose that $\bigvee\{h_*(v) \mid v \in I\} = p$ and let us prove that I coconverges to p. Let $x \in M$ such that $x \vee p = 1$, this means that $\{x\} \vee (\bigvee\{h_*(v) \mid v \in I\}) = 1_M$, so that the set $\{x\} \cup \bigvee\{h_*(v) \mid v \in I\}$ is a cover of M. Therefore, since M is compact, $\exists k \in \mathbb{N}$ such that $v_1, v_2, \ldots, v_k \in I$ and $x \vee h_*(v_1) \vee h_*(v_2) \vee \ldots \vee h_*(v_k) = 1_M$ and then

$$\begin{aligned} 1_L &= h(1_M) \\ &= h(x \vee h_*(v_1) \vee h_*(v_2) \vee \ldots \vee h_*(v_k)) \\ &= h(x) \vee hh_*(v_1) \vee hh_*(v_2) \vee \ldots \vee hh_*(v_k) \\ 1_L &= h(x) \vee v_1 \vee v_2 \vee \ldots \vee v_k \quad \text{(because since } h \text{ is onto, then } hh_* = id_M). \end{aligned}$$

Hence I coconverges to p.

□

4.1.16 Corollary. *Let L be a completely regular frame, I a proper ideal $J \in \Sigma\beta L$, then I coconverges to J if and only if $J \subseteq I$.*

Proof.

- If I coconverges to J, then $\bigvee_{s \in I} r(s) = J$ by Proposition (4.1.15). Suppose that $a \in J$, then $a \in \bigvee_{s \in I} r(s) = \bigcup_{s \in I} r(s)$ so that $\exists v \in I$ such that $a \in r(v) = \{x \in L \mid x \prec\!\!\prec v\}$, which means that $a \prec\!\!\prec v$. Since the relation $\prec\!\!\prec$ interpolates, $\exists u \in r(v) \subseteq I$ such that $a \prec\!\!\prec u \prec\!\!\prec v$, and then $a \in I$ because I is an ideal containing u. Hence $J \subseteq I$.

- If $J \subseteq I$, then $\forall u \in J, \exists v \in J \subseteq I$ such that $u \lll v$, because $J \in \beta L$. Therefore $\forall u \in J, \exists v \in I$ such that $u \in r(v)$, and then $J \subseteq \bigcup_{s\in I} r(s) = \bigvee_{s\in I} r(s)$. Since βL is regular, then it is T_1, and since J is prime in βL, it is a maximal element in βL and then $J \subseteq \bigcup_{s\in I} r(s) = \bigvee_{s\in I} r(s)$ implies that $\bigvee_{s\in I} r(s) = J$ or $\bigvee_{s\in I} r(s) = 1_{\beta L}$. Suppose that $\bigvee_{s\in I} r(s) = 1_{\beta L}$, therefore $\{r(s) \mid s \in I\}$ is a cover of βL and then $\exists p \in \mathbb{N}$ such that $s_1, s_2, \ldots, s_p \in I$ and $r(s_1) \vee r(s_2) \vee \ldots \vee r(s_p) = 1_{\beta L}$. Therefore

$$\begin{aligned} 1_L &= j(1_{\beta L}) \\ &= j(r(s_1) \vee r(s_2) \vee \ldots \vee r(s_p)) \\ &= jr(s_1) \vee jr(s_2) \vee \ldots \vee jr(s_p) \quad \text{(because } j \text{ is onto)} \\ 1_L &= s_1 \vee s_2 \vee \ldots \vee s_p. \end{aligned}$$

  Since $s_1, s_2, \ldots, s_p \in I$ and I is an ideal, then $1_L = s_1 \vee s_2 \vee \ldots \vee s_p \in I$ so that $I = L$ but this contradicts the fact that I is a proper ideal. It follows that

$$\bigcup_{s\in I} r(s) = \bigvee_{s\in I} r(s) = J.$$

  Hence by Proposition (4.1.15), I coconverges to J.

$\square$

4.2 Applications of localic convergence

In this section we use the convergence notion we have defined in the previous one to characterise sharp points and some types of locales.

Characterising sharp points.

4.2.1 Definition. For any point $I \in \Sigma\beta L$, we set

$$\mathbf{U}^I = \{\alpha \in \mathcal{R}L \mid r_L(\alpha) \subseteq I\} \qquad \text{and} \qquad \mathbf{A}^I = \{\mathfrak{c}_{r_L[L]}(r_L(\alpha)) \mid \alpha \in \mathbf{U}^I\}.$$

It is shown in [20] that $\mathbf{U}^I$ is a maximal ideal of $\mathcal{R}L$.

4.2.2 Definition. [13] A **sharp point** in a frame L is an element $I \in \Sigma\beta L$ such that $\forall \alpha \in \mathcal{R}L, r_L(\alpha) \subseteq I$ implies that $\alpha \in I$. It is also equivalent to saying that

$$I = \mathbf{U}^I = \{\alpha \in \mathcal{R}L \mid r(\alpha) \subseteq I\}.$$

4.2.3 Observation. *Let L be a frame, $a \in L$ and $p \in \Sigma L$, then $a \leq p$ implies that $a \vee p \neq 1$. In fact, if $a \leq p$, then $a \vee p = p \neq 1$. This statement is also equivalent to saying that $a \vee p = 1$ implies that $a \nleq p$.*

4.2.4 Remark. *Let L be a completely regular frame such that $u \in L$ and $I \in \beta L$ $r(u) \prec I$, then $u \in I$.*

In fact, since j is the right adjoint of r, saying that $r(u) \prec I$ means that

$$1_{\beta L} = r(u)^* \vee I = r(u^*) \vee I,$$

it follows that $u^* \vee (\bigvee I) = j(r(u^*) \vee j(I)) = j(r(u^*) \vee I) = j(1_{\beta L}) = 1_L$ because j is onto and then $j(r(u^*)) = u^*$. Therefore $u^* \vee (\bigvee I) = 1_L$, which means that $u^* \vee (\bigvee I) = \bigvee (I \cup \{u^*\}) = 1_L$ so that $I \cup \{u^*\}$ is a cover of L. Since L is completely regular, it is compact and then, $p \in \mathbb{N}$ such that $a_1, a_2, \ldots a_p \in I$ and $u^* \vee a_1 \vee a_2 \vee \ldots \vee a_p = 1_L$. We set $a = a_1 \vee a_2 \vee \ldots \vee a_p$, then $a \in I$ because I is an ideal in L. Therefore $u^* \vee a = 1_L$ so that $u \prec a$ an then $u \leq a$, it then follows that $u \in I$ because I is an ideal containing a.

4.2.5 Property. *If $\alpha, \beta \in \mathcal{R}L$ then $r(\alpha \vee_L \beta) = r(\alpha) \vee_{\beta L} r(\beta)$.*

4.2.6 Remark. *If $\psi \colon M \to N$ is a dense onto frame homomorphism, we have $\psi[M] = N$. Therefore the nucleus associated with ψ is $\nu_\psi = \psi_* \psi \colon M \to M$ and then*

$$S_{\nu_\psi} = \psi_* \psi[M] = \psi_*[N] \subseteq M$$

is the sublocale associated with ν_ψ. Then the open sublocales in S_{ν_ψ} are given by

$$\mathfrak{o}_{S_{\nu_\psi}}(t) = \mathfrak{o}_{S_{\nu_\psi}}(\nu_\psi(t)) = S_{\nu_\psi} \cap \mathfrak{o}(t) \text{ with } t \in M.$$

This is to say that $\mathfrak{o}_{\psi_*[N]}(t) = \mathfrak{o}_{\psi_*[N]}(\psi_*\psi(t)) = \psi_*[N] \cap \mathfrak{o}(t), \forall t \in M$. By using the same way of thinking, we get that the closed sublocales in S_{ν_ψ} are given by

$$\mathfrak{c}_{\psi_*[N]}(t) = \mathfrak{c}_{\psi_*[N]}(\psi_*\psi(t)) = \psi_*[N] \cap \mathfrak{c}(t), \forall t \in M.$$

4.2.7 Proposition. *The following statements are equivalent for any point $J \in \Sigma\beta L$.*

1) *J is a sharp point.*

2) *Every prime filter on $r[L]$ which converges to J contains $\mathbf{A}^J$.*

3) *Every ultrafilter on $r[L]$ which converges to J contains $\mathbf{A}^J$.*

4) *Every Cartan filter on $r[L]$ which converges to J contains $\mathbf{A}^J$.*

Proof.

- Let J be a sharp point and $\mathcal{F}$ a prime filter on $r[L]$ which converges to J. Prove that $\mathbf{A}^J \subseteq \mathcal{F}$. Assume that $s \in A^J$ then, $s = \mathfrak{c}_{r[L]}(r(\alpha))$, for some $\alpha \in \mathbf{U}^J$ ($\alpha \in$ coz$L$), which implies that $\alpha \in J = U^J$ because J is a sharp point. Since $J \in \Sigma\beta L \subseteq \beta L$, J is a completely regular ideal and then $\exists \beta \in J$ such that $\alpha \prec\!\!\prec \beta$, then $\exists \gamma \in J \cap$ coz$L$ such that $\alpha \wedge \gamma = 0_L$ and $\beta \vee \gamma = 1_L$. We get:

  $0_{r[L]} = r(0_L) = r(\alpha \wedge_L \gamma) = r(\alpha) \wedge_{\beta L} r(\gamma)$ and $r(\beta) \sqcup r(\gamma) = r(\beta \vee_{r[L]} \gamma) = 1_{r[L]} = r(1_L)$.

This is because $r[L]$ is a sublocale of βL then the join of two elements in $r[L]$ is defined by $r(\beta \vee \gamma) = r(\beta) \sqcup r(\gamma)$. Therefore

$$r[L] = \mathfrak{c}_{r[L]}(0_{r[L]}) = \mathfrak{c}_{r[L]}(r(\alpha \wedge_L \gamma)) = \mathfrak{c}_{r[L]}(r(\alpha) \wedge_{\beta L} r(\gamma)) = \mathfrak{c}_{r[L]}(r(\alpha)) \vee_{\beta L} \mathfrak{c}_{r[L]}(r(\gamma)).$$

Since $r(\beta) \leq J$ and $r(\gamma) \vee r(\beta) = 1_{\beta L}$, we have that $r(\gamma) \nleq J$. In fact, $r(\gamma) \leq J$ implies that $1_{\beta L} = r(\gamma) \vee r(\beta) \leq J$, which means that $1_{\beta L} \in J$ and which contradicts the fact that $J \in \Sigma\beta L$. Then $J \in \mathfrak{o}_{r[L]}(\gamma)$ and by convergence of $\mathcal{F}$, $\exists F \in \mathcal{F}$such that $F \subseteq \mathfrak{o}_{r[L]}(\gamma)$. Therefore $\mathfrak{o}_{r[L]}(\gamma) \in \mathcal{F}$ and then $\mathfrak{c}_{r[L]}(\gamma) \notin \mathcal{F}$ because $\mathfrak{o}_{r[L]}(\gamma) \cap \mathfrak{c}_{r[L]}(\gamma) = 0 \notin \mathcal{F}$. We have $\mathfrak{c}_{r[L]}(r(\alpha)) \vee_{\beta L} \mathfrak{c}_{r[L]}(r(\gamma)) = r[L] \in \mathcal{F}$ and since $\mathcal{F}$ is prime and $\mathfrak{c}_{r[L]}(\gamma) \notin \mathcal{F}$, this implies that $\mathfrak{c}_{r[L]}(r(\alpha)) \in \mathcal{F}$. Then $\mathbf{A}^J \subseteq \mathcal{F}$ and hence 1) implies 2).

- Since every ultrafilter on a locale (sublocale) is prime, then 2) implies 3).

- Assume that every ultrafilter on $r[L]$ which converges to J contains $\mathbf{A}^J$. Let $\mathcal{F}$ be a Cartan filter on $r[L]$ which converges to J, we prove that $\mathbf{A}^J \subseteq \mathcal{F}$. Since $\mathcal{F}$ is a filter in Sub($r[L]$)$\subseteq \mathcal{S}\ell(r[L]$, we have $0 \notin \mathcal{F}$ and $\forall F_1, F_2 \in \mathcal{F}, F_1 \wedge F_2 \in \mathcal{F}$ and then $\mathcal{F}$ generates a proper filter $\mathcal{F}_1$ in $\mathcal{S}\ell(r[L])$. Then, $\mathcal{F}_1$ is contained in an ultrafilter $\mathcal{G}$ in $\mathcal{S}\ell(r[L])$ so that $\mathcal{F} \subseteq \mathcal{F}_1 \subseteq \mathcal{G}$. Since $\mathcal{F}$ converges to J, $\forall a \in L, p \in \mathfrak{o}(a)$ implies that $\exists F \in \mathcal{F} \subseteq \mathcal{G}$ such that $F \subseteq \mathfrak{o}(a)$, then $\mathcal{G}$ also converges to p and then by hypothesis, $\mathbf{A}^J \subseteq \mathcal{G}$. We set

$$\mathcal{G}^c = \mathcal{G} \cap \text{Sub}(r[L]).$$

Let us prove that $\mathcal{G}^c$ is a proper filter on Sub($r[L]$):

  - O $\notin \mathcal{G}^c$ because O $\notin \mathcal{G}$.
  - If $S_1, S_2 \in \mathcal{G}^c$, then $S_1, S_2 \in \mathcal{G} \cap \text{Sub}(r[L])$ so that $S_1 \wedge S_2 \in \mathcal{G}$ and

$$(S_1 \wedge S_2)^c = S_1^c \vee S_2^c \in \mathcal{S}\ell(r[L]),$$

    therefore $S_1 \wedge S_2 \in \text{Sub}(r[L])$. It follows that $S_1 \wedge S_2 \in \mathcal{G}^c$.
  - If $S \in \mathcal{G}^c$ and

$$T \in \mathcal{S}\ell(r[L])$$

    such that $S \subseteq T$, therefore $T \in \mathcal{G}$ and $T^c \subseteq S^c \in \mathcal{S}\ell(r[L])$. It follows that $T^c \in \mathcal{S}\ell(r[L])$ so that $T \in \text{Sub}(r[L])$. Therefore, $T \in \mathcal{G}^c$.

Furthermore, we have $\mathcal{F} \subseteq \mathcal{G} \cap \text{Sub}(r[L]) = \mathcal{G}^c$. Therefore $\mathcal{G}^c$ is a proper filter in Sub($r[L]$) containing $\mathcal{F}$, therefore $\mathcal{G}^c = \mathcal{F}$ because $\mathcal{F}$ is maximal in terms of inclusion of all filters in Sub($r[L]$). Since every element in A^J is a closed sublocale in $r[L]$ this means that $A^J \subseteq \text{Sub}(r[L])$ then $A^J \subseteq \text{Sub}(r[L]) \cap \mathcal{G} = \mathcal{G}^c = \mathcal{F}$ so that $A^J \subseteq \mathcal{F}$. Hence 3) implies 4).

- Assume that every Cartan filter on $r[L]$ which converges to J contains $\mathbf{A}^J$. We must show that J is a sharp point. Suppose that $\mathbf{U}^J \nsubseteq J$, this means that $\exists \alpha \in \mathcal{R}L$ such that $r(\alpha) \subseteq J$ and $\alpha \notin J$. Therefore $r(\alpha)^* \vee J \neq 1_{\beta L}$. In fact, if $r(\alpha)^* \vee J = 1_{\beta L}$, then $r(\alpha) \prec J$ and by Remark(4.2.4) this implies that $\alpha \in J$ and it contradicts the fact that $\alpha \notin J$. It follows from Observation (4.2.3) that $r(\alpha)^* \leq J$, which means that

$J \in \uparrow (r(\alpha)^*) = \mathfrak{c}_{r[L]}(r(\alpha)^*) = \overline{\mathfrak{o}_{r[L]}(r(\alpha))}$.

Before going ahead with our proof, let us first observe some useful facts:

4.2.8 Observation. *Let $\theta \colon M \to N$ be a dense onto frame homomorphism and $p \in \Sigma M$, then the following statements are true:*

- *If $p \in \overline{\mathfrak{o}(m)}$ for some $m \in M$, then $\forall b \in M, p \in \mathfrak{o}(b)$ implies that $\mathfrak{o}(m) \cap \mathfrak{o}(b) \neq O$. In fact, if $\mathfrak{o}(m) \cap \mathfrak{o}(t) = O$ for some $t \in M$, then $\mathfrak{o}(m \wedge t) = \mathfrak{o}(m) \cap \mathfrak{o}(t) = O$, which means that $m \wedge t = 0$, so that $t \leq m^*$. Suppose that $m^* \leq p$, then $m^* \vee p = p$ and since $t \leq m^*$, we get that $t \vee p \leq m^* \vee p$. Since $p \in \mathfrak{o}(t)$, it follows from (4.2.3) that $t \vee p = 1$, then $1 = t \vee p \leq m^* \vee p = p$ and then $p = 1$, which contradicts the fact that $p \neq 1$. Therefore $m^* \nleq p$ and this contradicts the fact that $p \in \overline{\mathfrak{o}(m)} = \overline{\mathfrak{c}(m^*)} = \uparrow m^* = \{x \in M \mid x \leq m^*\}$. Hence $\mathfrak{o}(m) \cap \mathfrak{o}(b) \neq O$.*
- *If $p \in \Sigma M$ and $p \in \overline{\mathfrak{o}(m)}$ for some $m \in M$, consider the set*

$$\mathcal{V} = \{\mathfrak{o}_{\theta_*[N]}(\theta_*\theta)(m \wedge t) \mid t \in M \text{ and } p \in \mathfrak{o}(t)\}.$$

Then $\mathcal{V}$ generates a proper Cartan filter on $\theta_*[N]$. In fact:

  - *If $O \in \mathcal{V}$, then $\exists s \in M$ such that $p \in \mathfrak{o}(s)$ and $O = \mathfrak{o}_{\theta_*[N]}(\theta_*\theta)(m \wedge s)$, which means by Remark (4.2.6) that*

$$O = \mathfrak{o}_{\theta_*[N]}(\theta_*\theta)(m \wedge s) = \theta_*[N] \cap \mathfrak{o}(m \wedge s) = \theta_*[N] \cap (\mathfrak{o}(m) \cap \mathfrak{o}(s)).$$

  This is not possible because an open sublocale meets a sublocale if and only if it meets its closure. Hence $O \neq \mathfrak{o}_{\theta_*[N]}(\theta_*\theta)(m \wedge s)$ and then $O \notin \mathcal{V}$.
  - *If $v_1, v_2 \in \mathcal{V}$, then $\exists s, t \in M$ such that*

$$p \in \mathfrak{o}(s) \cap \mathfrak{o}(t) = \mathfrak{o}(s \wedge t),$$

  $v_1 = \mathfrak{o}_{\theta_*[N]}(\theta_*\theta)(m \wedge t)$ and $v_2 = \mathfrak{o}_{\theta_*[N]}(\theta_*\theta)(m \wedge s)$ so that by Remark (4.2.6),

$$\begin{aligned} v_1 \cap v_2 &= (\theta_*[N] \cap \mathfrak{o}(m \wedge s) \cap (\theta_*[N] \cap \mathfrak{o}(m \wedge s)) \\ &= (\theta_*[N] \cap \mathfrak{o}(m) \cap \mathfrak{o}(s)) \cap (\theta_*[N] \cap \mathfrak{o}(m) \cap \mathfrak{o}(t)) \\ &= \theta_*[N] \cap (\mathfrak{o}(s) \cap \mathfrak{o}(t)) \cap \mathfrak{o}(m) \\ v_1 \cap v_2 &= \mathfrak{o}_{\theta_*[N]}(\theta_*\theta)(m \wedge (s \wedge t)) \in \mathcal{V}. \end{aligned}$$

  Therefore $\mathcal{V}$ generates a Cartan filter $\mathcal{F}$ on $\theta_*[N]$.
  Let us prove that $\mathcal{F}$ converges to p. Let w be in M such that $p \in \mathfrak{o}(w)$, therefore we have

$$\mathfrak{o}_{\theta_*[N]}(\theta_*\theta)(m \wedge w) \subseteq \mathfrak{o}_{\theta_*[N]}(\theta_*\theta)(w) = \theta_*[N] \cap \mathfrak{o}(w) \subseteq \mathfrak{o}(w) \text{ because } m \wedge w \leq w.$$

  It follows that $\mathfrak{o}(w)$ contains $\mathfrak{o}_{\theta_*[N]}(\theta_*\theta)(m \wedge w) \in \mathcal{V} \subseteq \mathcal{F}$, therefore $\mathcal{F}$ converges to p.

Now Let us go back to our main proof. By letting $\theta = \bigvee = j\colon \beta L \to L$ and $p = J \in \Sigma\beta L$, then

$$\mathcal{V} = \{\mathfrak{o}_{r[L]}(rj(I \wedge r(\alpha))) \mid I \in \beta L \text{ and } J \in \mathfrak{o}(I)\}.$$

Since the Cartan Filter $\mathcal{F}$ generated by $\mathcal{V}$ is a filter in the boolean algebra Sub$(r[L])$, there is an ultrafilter $\mathcal{G}$ on $r[L]$ such that $\mathcal{F} \subseteq \mathcal{G}$. Since $\mathcal{F}$ converges to $J, \forall I \in \beta L$ such that $J \in \mathfrak{o}(I), \exists S \in \mathcal{F} \subseteq \mathcal{G}$ such that $S \subseteq \mathfrak{o}(I)$ then, $\mathcal{G}$ also converges to J. Therefore by hypothesis $\mathbf{A}^J \subseteq \mathcal{G}$. Now, assume that $I = 1_{\beta L}$, then

$$\mathfrak{o}_{r[L]}(rj(1_{\beta L} \wedge r(\alpha))) = \mathfrak{o}_{r[L]}(rj(r(\alpha))) = \mathfrak{o}_{r[L]}(r(\alpha)) \in \mathcal{V} \ \subseteq \mathcal{F}.$$

This is because j is onto, then $rj(r(\alpha)) = r(\alpha)$. But, $\alpha \in \mathbf{U}^J$ implies that

$$\mathfrak{c}_{r[L]}(r(\alpha)) \in \mathbf{A}^J \subseteq \mathcal{F}$$

so that $\mathfrak{c}_{r[L]}(r(\alpha)) \cap \mathfrak{o}_{r[L]}(r(\alpha)) = \mathsf{O} \in \mathcal{F}$, this is a contradiction because $\mathcal{F}$ is a filter and then $\mathsf{O} \notin \mathcal{F}$. Therefore $\mathbf{U}^J \subseteq J$ and by maximality of $\mathbf{U}^J \in \mathcal{R}L$, it follows that $\mathbf{U}^J = J$, which means that J is a sharp point. Hence 4) implies 1).

$\square$

Characterising compact locales.

4.2.9 Proposition. *[13, Proposition 4.2] Let L be a locale. Then the following statements are equivalent:*

1) *L is compact.*
2) *Every filter on L is clustered.*
3) *Every Cartan filter on L is clustered.*
4) *Every Cartan ultrafilter on L is convergent.*
5) *Every prime filter on L is convergent.*
6) *Every ultrafilter on L is convergent.*

Proof.

- Let L be a compact frame and $\mathcal{F}$ be a filter on L. If $\mathcal{F}$ does not cluster, then

$$P = \bigcap\{\overline{F} \mid F \in \mathcal{F}\} = \mathsf{O}.$$

In fact, if $P \neq 0$, then by Lemma (1.2.15) $\exists p \in P$ and this would mean that $\mathcal{F}$ clusters at p and this contradicts our hypothesis.
For every $F \in \mathcal{F}$ we set $s_F = \bigwedge F$, then $\overline{F} = \uparrow s_F = \mathfrak{c}(s_F)$. Let us prove that

$$\bigvee\{s_F \mid F \in \mathcal{F}\} = 1_L.$$

To do this, we will prove that $\bigvee\{\mathfrak{o}(s_F) \in \mathcal{S}\ell(L) \mid F \in \mathcal{F}\} = L$. In fact, we have

$$\bigvee\{s_F \mid F \in \mathcal{F}\} = 1_L, \text{ if and only if } \mathfrak{c}(\bigvee\{s_F \mid F \in \mathcal{F}\}) = \mathfrak{c}(1_L) = \mathsf{O}$$

$$\text{if and only if } \bigcap\{\mathfrak{c}(u_F) \mid F \in \mathcal{F}\} = \mathsf{O},$$

which is equivalent to

$$L \setminus (\bigcap\{\mathfrak{c}(u_F) \mid F \in \mathcal{F}\}) = \bigvee_{\mathcal{S}\ell(L)} \{\mathfrak{o}(s_F) \mid F \in \mathcal{F}\} = L \setminus \mathsf{O} = L.$$

Let $L \setminus S$ be the complement of an element $S \in \mathsf{Sub}(L)$, we have :

$$\begin{aligned} L &= L \setminus \mathsf{O} \\ &= L \setminus \bigcap\{\overline{F} \mid F \in \mathcal{F}\} \\ &= \bigvee_{\mathcal{S}\ell(L)} \{L \setminus \overline{F} \mid F \in \mathcal{F}\} \\ &= \bigvee_{\mathcal{S}\ell(L)} \{L \setminus \mathfrak{c}(s_F) \mid F \in \mathcal{F}\} \\ L &= \bigvee_{\mathcal{S}\ell(L)} \{\mathfrak{o}(s_F) \mid F \in \mathcal{F}\}. \end{aligned}$$

Since L is compact and $\bigvee\{s_F \mid F \in \mathcal{F}\} = 1_L$, there exists $k \in \mathbb{N}$ such that

$$\bigvee\{s_{F_1}, s_{F_2}, \ldots s_{F_k}\} = 1_L$$

for finitely many $F_1, F_2, \ldots, F_k \in \mathcal{F}$, so that

$$\overline{F_1} \cap \overline{F_2} \cap \ldots \cap \overline{F_k} = \mathfrak{c}(s_{F_1} \vee s_{F_2} \vee \ldots \vee s_{F_k}\}) = \mathfrak{c}(1_L) = \mathsf{O}.$$

This is a contradiction because since $F_1, F_2, \ldots, F_k \in \mathcal{F}$ and $F_i \subseteq \overline{F_i}$ for $1 \leq i \leq k$, we have $\overline{F_i} \in \mathcal{F}$ for $1 \leq i \leq k$, implying that $\overline{F_1} \cap \overline{F_2} \cap \ldots \cap \overline{F_k} \in \mathcal{F}$. Hence 1) implies 2) .

- Suppose that every filter on L is clustered and let $\mathcal{F}$ be a cartan filter on L, this means a filter on $\mathsf{Sub}(L)$. Since $0 \notin \mathcal{F}$ and $\mathcal{F}$ is closed under the finite meet, $\mathcal{F}$ generates a proper filter $\mathcal{G}$ on L, which is clustered by hypothesis, so that $\exists p \in \Sigma L$ such that every neighbourhood of p meets every element of $\mathcal{G}$ and then every neighbourhood of p meets every element of $\mathcal{F}$ in particular, because $\mathcal{F} \subseteq \mathcal{G}$ and so $\mathcal{F}$ is clustered. Hence 2) implies 3) .

- Assume that every Cartan filter on L is convergent and let $\mathcal{F}$ be a Cartan ultrafilter on L. Then by hypothesis, $\mathcal{F}$ clusters to some $p \in \Sigma L$. This means that $\forall a \in L, p \in \mathfrak{o}(a)$ implies that $\mathfrak{o}(a) \cap S \neq \mathsf{O}, \forall S \in \mathcal{F}$. Assume that $p \in \mathfrak{o}(a)$ for some $a \in L$, then by Remark (1.2.11) $\mathfrak{o}(a) \in \mathsf{Sub}(L)$, which is a Boolean algebra. Then by Proposition(1.1.38), we have either $\mathfrak{o}(a) \in \mathcal{F}$ or $L \setminus \mathfrak{o}(a) = \mathfrak{c}(a) \in \mathcal{F}$. If $\mathfrak{c}(a) \in \mathcal{F}$, then $\mathfrak{o}(a) \cap \mathfrak{c}(a) \neq \mathsf{O}$, which is a contradiction. Therefore $\mathfrak{o}(a) \in \mathcal{F}$ and $\mathfrak{o}(a) \subseteq \mathfrak{o}(a)$ so that $\mathcal{F}$ is convergent and hence 3) implies 4).

- Suppose that $\mathcal{F}$ is a prime filter on L, we set $\mathcal{F}^c = \mathcal{F} \cap \mathsf{Sub}(L)$. Let us prove that $\mathcal{F}^c$ is a Cartan filter on L.

  - $0 \notin \mathcal{F}$ implies that $0 \notin \mathcal{F}^c$.
  - If $S, T \in \mathcal{F}^c$, then $S, T \in \mathcal{F}$ and $S, T \in \mathsf{Sub}(L)$ so that $S \cap T \in \mathcal{F}$ and
  $$L \setminus (S \cap T) = (L \setminus S) \vee (L \setminus T).$$
  Therefore $S \cap T \in \mathcal{F}^c$.
  - If $S \in \mathcal{F}^c$ and $T \in \mathcal{S}\ell(L)$ such that $S \subseteq T$, then $S \in \mathcal{F} \cap \mathsf{Sub}(L)$ so that $T \in \mathcal{F}$ and $L \setminus T \subseteq L \setminus S$. Therefore $T \in \mathcal{F}^c$.

  Hence $\mathcal{F}^c$ is a Cartan filter on L.
  Now let us prove that $\mathcal{F}^c$ is maximal. Since $\mathsf{Sub}(L)$ is a boolean algebra, by Proposition (1.1.38), it is equivalent to prove that $\forall S \in \mathsf{Sub}(L)$, either $S \in \mathcal{F}^c$ or $L \setminus S \in \mathcal{F}^c$, property which is always true. In fact, if $S \in \mathsf{Sub}(L)$, then $S \vee L \setminus S) = 1_{\mathcal{S}\ell(L)} = L \in \mathcal{F}^c \subseteq \mathcal{F}$, which implies that either $S \in \mathcal{F}^c$ or $L \setminus S \in \mathcal{F}^c$ because $\mathcal{F}$ is a prime filter on L. Therefore $\mathcal{F}^c$ is an ultrafilter in $\mathsf{Sub}(L)$ which is convergent by hypothesis. The convergence of $\mathcal{F}^c$ means that $\exists p \in \Sigma L$ such that $\forall a \in L, p \in \mathfrak{o}(a)$ implies that $\exists F \in \mathcal{F}^c \subseteq \mathcal{F}$ such that $F \subseteq \mathfrak{o}(a)$. Hence $\mathcal{F}$ is convergent so that 4) implies 5).

- 5) obviously implies 6) because every ultrafilter is prime.

- Suppose that every ultrafilter on L converges. Assume that L is not compact. This means that there is a cover U of L which does not have any finite subcover. Therefore for every finite $M \subseteq U$, $\bigvee M \neq 1_L$ so that
$$\bigcap_{m \in M} \mathfrak{c}(m) = \mathfrak{c}(\bigvee M) \neq \mathfrak{c}(1_L) = \mathsf{O}.$$
We set $\mathcal{A} = \{\bigcap_{t \in T} \mathfrak{c}(t) \mid T \text{ is a finite subset of } U\}$. Let us prove that $\mathcal{A}$ is a base for a proper filter in L.

  - By definition, $\mathsf{O} \notin \mathcal{A}$.
  - If $M, N \in \mathcal{A}$, then $M = \bigcap_{a \in A} \mathfrak{c}(a)$ and $N = \bigcap_{b \in B} \mathfrak{c}(b)$ for some finite subsets $A, B \subseteq U$ so that
  $$\begin{aligned} M \cap N &= \bigcap_{a \in A} \mathfrak{c}(a) \cap \bigcap_{b \in B} \mathfrak{c}(b) \\ &= \mathfrak{c}(\bigvee_{a \in A} a) \cap \mathfrak{c}(\bigvee_{b \in B} b) \\ &= \mathfrak{c}(\bigvee_{a \in A} a \vee \bigvee_{b \in B} b) \\ &= \mathfrak{c}(\bigvee_{t \in A \cup B} t) \\ M \cap N &= \bigcap_{t \in A \cup B} \mathfrak{c}(t). \end{aligned}$$

Since A and B are finite subsets of U, so is $A \cup B$. Consequently, $M \cap N \in \mathcal{A}$ and hence $\mathcal{A}$ generates a proper filter $\mathcal{G}$ on L which is a filterbase for an ultrafilter $\mathcal{F}$ on L. Since $\bigvee U = 1$, then $\bigvee U \nleq p$ so that $\exists u \in U$ such that $u \nleq p$. It follows that $p \in \mathfrak{o}(u)$ and since the filter $\mathcal{F}$ converges by hypothesis, $\exists F \in \mathcal{F}$ such that $F \subseteq \mathfrak{o}(u)$ and then $\mathfrak{o}(u) \in \mathcal{F}$. On the other hand, the subset $\{u\} \subseteq U$ is finite, so $\mathfrak{c}(u) \in \mathcal{A} \subseteq \mathcal{G} \subseteq \mathcal{F}$. Finally we get that $\mathfrak{o}(u) \wedge \mathfrak{c}(u) = \mathsf{O} \in \mathcal{F}$, which contradicts the fact that $\mathsf{O} \notin \mathcal{F}$. Hence 6) implies 1).

□

4.2.10 Proposition. *Let $h\colon M \to L$ be an extension of a locale L and let I be an ideal in M. The ideal I coclusters at a point $p \in \Sigma M$ if and only if its associated filter $\mathcal{F}_I$ clusters.*

Proof.

- Assume that I coclusters at $p \in \Sigma M$ and let us prove that $p \in \bigcap\{\overline{F} \mid F \in \mathcal{F}_I\}$. Let $F \in \mathcal{F}_I$ and we set $s = \bigwedge F$ so that $\overline{F} = \uparrow (\bigwedge F) = \mathfrak{c}(s)$. Since $F \subseteq \overline{F}^{h_*[L]}$ and $F \in \mathcal{F}_I$, then
$$\overline{F}^{h_*[L]} = \overline{F} \cap h_*[L] = \mathfrak{c}_{h_*[L]}(h_*h(s)) \in \mathcal{F}_I,$$
where $\mathcal{F}_I = \{S \in \mathcal{S}\ell(h_*[L]) \mid \mathfrak{c}_{h_*[L]}(h_*(a)) \subseteq S \text{ for some } a \in I\}$. Since
$$\mathfrak{c}_{h_*[L]}(h_*h(s)) \in \mathcal{F}_I,$$
then $h(s) \in I_{\mathcal{F}_I} = I$. The fact that I coclusters at p implies that
$$\bigvee\{h_*(u) \mid u \in I\} \leq p,$$
so that $h_*(h(s)) \leq \bigvee\{h_*(u) \mid u \in I\} \leq p$, which means that $p \in\uparrow h_*h((s)) = \mathfrak{c}(h_*h(s))$. Furthermore, $s \leq h_*h(s)$, which means that $\mathfrak{c}(h_*h(s)) \subseteq \mathfrak{c}(s) = \overline{F} \in \mathcal{F}$. Therefore $p \in \overline{F}$ and hence $p \in \bigcap\{\overline{F} \mid F \in \mathcal{F}_I\}$ and then $\mathcal{F}_I$ clusters.
- Conversely, assume that $\mathcal{F}_I$ clusters at some point $p \in \Sigma M$. Let $v \in I = I_{\mathcal{F}_I}$, then $\mathfrak{c}_{h_*[L]}(h_*(v)) \in \mathcal{F}_I$. Since $\mathcal{F}_I$ clusters at p, we have that $p \in \bigcap\{\overline{F} \mid F \in \mathcal{F}_I\}$ so that $p \in \overline{\mathfrak{c}_{h_*[L]}(h_*(v))}$. We get :
$$p \in \overline{\mathfrak{c}_{h_*[L]}(h_*(v))} = \overline{\mathfrak{c}(h_*(v)) \cap h_*[L]} \subseteq \overline{\mathfrak{c}(h_*(v))} \cap \overline{h_*[L]} = \overline{\mathfrak{c}(h_*(v))} \cap M = \mathfrak{c}(h_*(v))$$
This is because $h_*[L]$ is a dense sublocale of M, (it means that $\overline{h_*[L]} = M$) and $\mathfrak{c}(h_*(v))$ is a closed sublocale, (it means that $\overline{\mathfrak{c}(h_*(v))} = \mathfrak{c}(h_*(v))$). It follows that
$$p \in \mathfrak{c}(h_*(v)) =\uparrow h_*(v),$$
which means that $h_*(v) \leq p, \forall v \in I$ and then $\bigvee\{h^*(v) \mid v \in I\} \leq p$. Hence I coclusters at p.

□

4.2.11 Remark. *If we consider a filter $\mathcal{F}$ on a locale M, then $\mathcal{F}$ clusters at some point $p \in \Sigma M$ if and only if its associated ideal $I_{\mathcal{F}}$ coclusters at p. The proof is obtained by using the same reasoning as in the proof of Proposition* (4.2.10).

This remark implies the following corollary.

4.2.12 Corollary. *A frame L is compact if and only if every ideal in it coclusters.*

Proof. By Proposition(4.2.9), L is compact if and only if every filter on L is clustered and by combining Remark (4.2.11) and Proposition(4.2.10), every filter in L clusters if and only if every ideal in L coclusters. $\square$

Conclusion

In this work, we were to study pointfree convergence along the lines of paper [13], which means convergence of filters on a locale. This turns out to be an interesting tool to characterise compact locales and sharp points in a frame. On the other hand, we reconciled this type of convergence with the previous work by [17] and [3]. To achieve this, we started by defining some concepts used along our dissertation. Then, we studied convergence and clustering in terms of filters in a frame by using covers and this turned out to fit in characterising compact, spatial and uniformly paracompact frames. We proved that classical filters can be identified as specific types of general filters in a frame. We extended our study of convergence on general filters. We introduced a new variant of compactness called $\mathbb{F}$-compactness for general filters and this gave us again some characterisations for compactness in frames to some extent. Although many authors tried to define the concept of clustering for general filters in a frame, we still do not have its right definition. So attempting to give this definition sounds good as further work. It will be also interesting if we could reformulate all the concepts of convergence, clustering and compactness used in this dissertation in terms of generalised sequences in a frame.

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